Convergence and Stability of Discrete Exterior Calculus for the Hodge Laplace Problem in Two Dimensions

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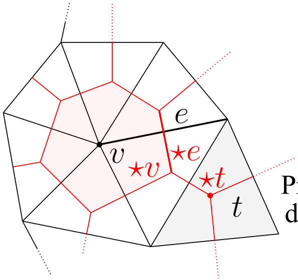
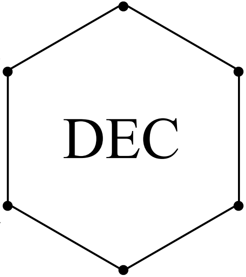
Discrete Exterior Calculus (DEC)



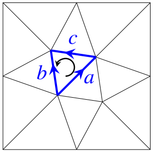
$$\begin{tikzcd}
C^0(X) \arrow[r, "d_0"] \arrow[d, shift left, "*_0"] & C^1(X) \arrow[r, "d_1"] \arrow[d, shift left, "*_1"] & C^2(X) \arrow[d, shift left, "*_2"] \\
D^2(\star X) \arrow[u, shift left, "*_0^{-1}"] & D^1(\star X) \arrow[l, "d_1^{\mathrm{dual}}"'] \arrow[u, shift left, "*_1^{-1}"] & D^0(\star X) \arrow[l, "d_0^{\mathrm{dual}}"'] \arrow[u, shift left, "*_2^{-1}"]
\end{tikzcd}$$

Separation of derivative and metric

Primal and dual mesh



Exterior derivative as coboundary



$a - b + c$

Discrete Hodge star via circumcentric duality

$$\frac{1}{|\star\sigma|}\int_{\star\sigma}\star\,\alpha = \frac{1}{|\sigma|}\int_{\sigma}\alpha$$

Wedge product as anti-symmetrized cup product

Naturality and simplicial maps

Primal and dual cochains notation

- Ω domain
- $\Lambda^k(\Omega)$ smooth forms
- X triangulation of Ω
- $C^k(X)$ space of real-valued k-cochains on X (discretized smooth forms)
- For $\hat{\alpha} \in \Lambda^k(\Omega)$, corresponding cochain $\alpha \in C^k(X)$ defined by

  $$\alpha(c) = \int_c \hat{\alpha}$$

  for chains $c \in C_k(X)$
- Evaluation notation: $\langle \alpha, c \rangle$ instead of $\alpha(c)$
- $\star X$ dual cell complex
- $D^k(\star X)$ space of dual cochains

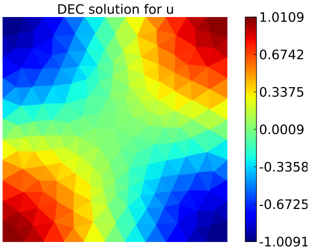
Hodge-Laplace problem mixed form (cochains version)

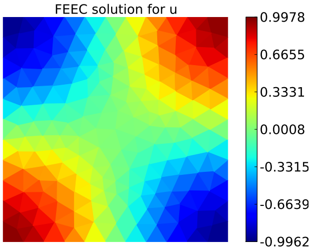
For $f \in C^k(X)$ find $(\sigma, u, p) \in C^{k-1}(X) \times C^k(X) \times \mathfrak{H}^k(X)$ such that,

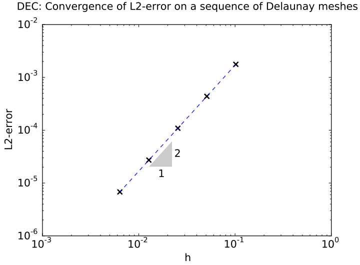
$$
\begin{aligned}
\langle \sigma, \tau \rangle - \langle u, d\tau \rangle &= 0 && \forall \tau \in C^{k-1}(X), \\
\langle d\sigma, v \rangle + \langle du, dv \rangle + \langle p, v \rangle &= \langle f, v \rangle && \forall v \in C^k(X), \\
\langle u, q \rangle &= 0 && \forall q \in \mathfrak{H}^k(X),
\end{aligned}
$$

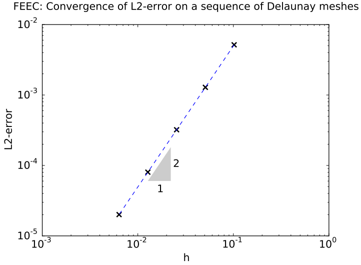
Numerical evidence of convergence ($k = 0$)

$u = \cos(\pi x)\cos(\pi y)$ on unit square









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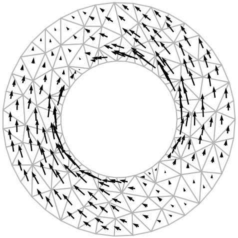
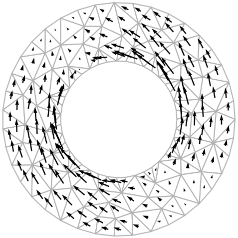
Numerical evidence of convergence ($k = 1$)

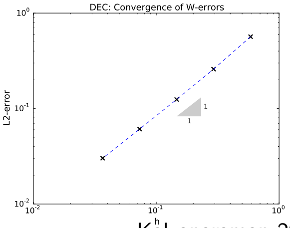
$u = (r - r_0)(r - r_1) \sin\theta \, dr, \quad h = 1/r \, d\theta$ on annulus

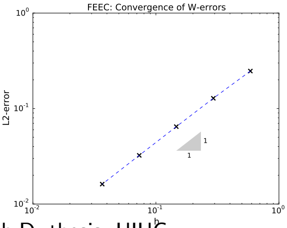
DEC solution for u



FEEC solution for u







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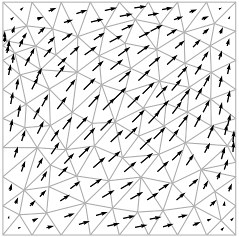
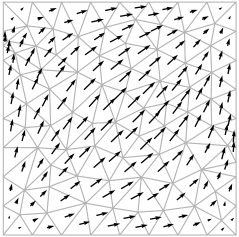
Numerical evidence of convergence ($k = 1$)

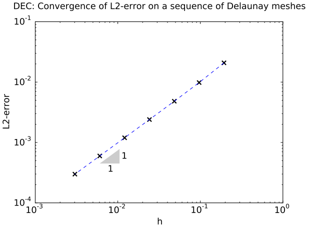
$$u = x\,(1-x)\,y^2\,(1-y)^2\,dx + x^2\,(1-x)^2\,y\,(1-y)\,dy \text{ on unit square}$$

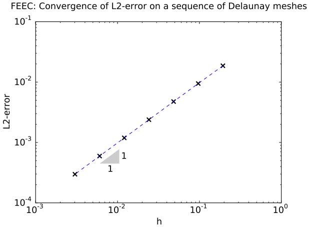
DEC solution for u



FEEC solution for u







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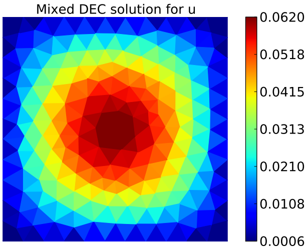
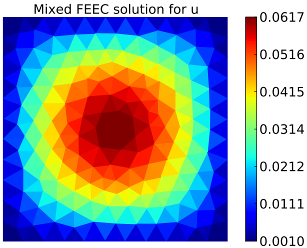
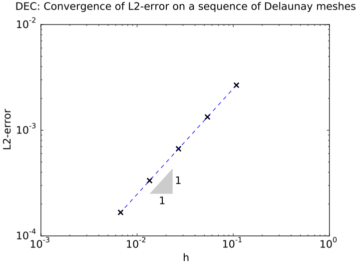
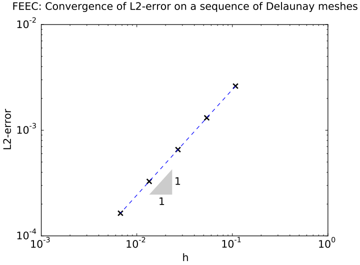
Numerical evidence of convergence ($k = 2$)

$u = x(1-x)y(1-y)dx \wedge dy$ on unit square









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DEC results implied by finite element results

For $k = 0$ DEC convergence for DEC-regular triangulations in 2D is implied by mass lumping results (Haugazeau and Lacoste 1993; Christiansen and Halvorsen 2011):

- Let M_1 be $-(\int_{T_-}\langle d\lambda_i, d\lambda_j\rangle d\mu + \int_{T+}\langle d\lambda_i, d\lambda_j\rangle d\mu)$ for internal edge $[i, j]$ shared by triangles T_- and T_+ and $-\int_T\langle d\lambda_i, d\lambda_j\rangle d\mu$ for boundary edge in triangle T
- For a shape regular family of triangulations of Ω the finite element solution for $k = 0$ is stable and convergent (Haugazeau and Lacoste 1993).

The DEC Hodge star $*_1^D$ is identical to the mass-lumped matrix M_1:

- For a triangle T, edge e_2, dual edge $\star e_2$:

$$-\int_T \langle d\lambda_0, d\lambda_1\rangle d\mu = \frac{\cos C}{2\sin C} = \frac{|\star e_2|}{|e_2|}$$

- Unique characterization: for non-degenerate Delaunay meshes, the lumped mass matrix M_1 (and thus $*_1^D$) is the *unique* diagonal matrix which yields the standard FE stiffness matrix (Bossavit and Kettunen 1999).

Other previous work

- Prior to this work (and simultaneous work by Guzmán and Potu 2025), convergence was only known for $k = 0$.
- Schulz and Tsogtgerel 2020: Proved convergence for $k = 0$ with essential Dirichlet BCs in general dimension.
- Their proof relied on norm equivalence for well-centered triangulations.
- Gap in Schulz and Tsogtgerel 2020: They claimed norm equivalence for shape-regular acute triangulations.

Counterexample

Consider a pair of triangles sharing edge e where sum of opposite angles $\to \pi$. If 1-cochain α is 1 on e and 0 elsewhere: $\|\alpha\|_D \to 0$ while $\|\alpha\|_F$ remains bounded.

Other previous work

- Mass Lumping: Used to relate FE to FDTD (Joly 2003) and Finite Volume (Baranger, Maitre, and Oudin 1996), but mostly for scalar Dirichlet problems.
- Covolume Methods: Nicolaides 1992 analyzed $k = 2$ (div-curl), but relied on specific orthogonal decompositions difficult to generalize.
- Other approaches: Adler et al. 2021 analyzed inf-sup for Maxwell's (div $B = 0$), using norm equivalence but missing the geometric dependency of the constant.

Two simultaneous but very different proofs

Convergence and Stability of Discrete Exterior Calculus for the Hodge Laplace Problem in Two Dimensions by C. Zhu, S. Christiansen, ANH, K. Hu (arXiv:2505.08966)

For appropriate hypothesis on triangulation families:

- DEC norm is equivalent to FEEC norm with constants independent of mesh refinement.
- DEC inner product is consistent with the FEEC inner product, with an explicit error estimate.
- We prove a Poincaré inequality for DEC, with respect to DEC-orthogonality.
- Relationship between DEC and FEEC harmonic forms.
- Using the above we prove convergence of the DEC solution by comparison with the FEEC solution.

A Framework for Analysis of DEC Approximations to Hodge-Laplacian Problems using Generalized Whitney Forms by Johnny Guzmán and Pratyush Potu (arXiv:2505.08934)

- Major tool: generalized Whitney forms (harmonic extension) on the dual cells
- Proof for well-centered flat domains in all dimensions

Mesh definitions

- Acute: $\theta < \pi/2$ for all angles θ.
- Uniformly acute: $\theta \leq \pi/2 - \delta_{\pi/2}$ for some fixed $\delta_{\pi/2} > 0$ for all angles θ.
- Boundary acute: $\theta_\partial < \pi/2$ for all angles θ_∂ opposite to boundary edges.
- Uniformly boundary acute: $\theta_\partial \leq \pi/2 - \delta_{\pi/2}$ for some fixed $\delta_{\pi/2} > 0$ for all angles θ_∂ opposite to boundary edges.
- Delaunay: $\theta_i + \theta_j \leq \pi$ for all pairs of angles θ_i, θ_j opposite to internal edges. Equivalently every circumcircle has no mesh vertex in its interior.
- Non-degenerate Delaunay: $\theta_i + \theta_j < \pi$ for all pairs of angles θ_i, θ_j opposite to internal edges. Equivalently every circumcircle has exactly 3 vertices on it.
- Uniformly Delaunay: $\theta_i + \theta_j \leq \pi - \delta_\pi$ for some fixed $0 < \delta_\pi < \pi$ for all pairs of angles θ_i, θ_j opposite to internal edges.

Mesh definitions (continued)

- Shape regular: $\theta \geq \delta_0$ for some fixed $\delta_0 > 0$ for all angles θ. Equivalently ratio of inscribed circle radius to circumcircle radius is bounded below.
- DEC regular: Shape regular and non-degenerate Delaunay.
- Curvature bounded: Angle defect at each vertex v is bounded below by $-N$, for some constant $N > 0$. Angle defect is $2\pi - \sum_i \theta_i$ where the sum is over all triangles i containing v and θ_i is the angle at v of triangle i.

Mesh definitions (continued)

For triangle T define

$$\delta_T := \min_{i \neq j \neq k} \frac{|\langle e_i, e_j \rangle|}{\|e_k\|^2}$$

- A family $\{X_h\}$ of triangulations of Ω being uniformly acute is equivalent to existence of a global constant $\delta > 0$ independent of h such that $\delta_T \geq \delta$ for all triangles T in the family

What is proved

For piecewise linear surface Ω:

- DEC and FEEC norms equivalent for uniformly acute families of triangulations of Ω
- DEC and FEEC norms equivalent for uniformly Delaunay uniformly boundary acute families of triangulations which are curvature bounded.
  - Curvature boundedness only needed for 0-forms
  - Only one side of norm equivalence is needed for stability and convergence
- For DEC regular families difference between DEC and FEEC inner products in terms of the mesh parameter h is estimated in terms of the $H\Lambda^k(\Omega)$ norm
- For DEC regular families two versions of DEC Poincaré equality are proved: in terms of FEEC and DEC norms, both using DEC orthogonality
- Comparison between DEC and FEEC harmonic forms.

What is proved (continued)

For $\Omega \subset \mathbb{R}^2$:

- Stability for DEC using FEEC norm and harmonic forms, and using DEC norm and harmonic forms
- Estimation of difference between DEC and FEEC solutions with natural boundary conditions in terms of FEEC norm and harmonic forms, and in terms of DEC harmonic forms

All these are for DEC regular families. Planarity needed due to FEEC machinery.
from Arnold, Falk, and Winther 2010.)

Discrete Hodge star

- Analogous to mass matrix in finite elements
- Defined by requiring
$$\frac{1}{|\star\sigma|}\langle *\alpha, \star\sigma\rangle = \frac{1}{|\sigma|}\langle \alpha, \sigma\rangle$$
  where $|\sigma|$ means measure of σ
- $*_k : C^k(X) \to D^{n-k}(\star X)$
- Discrete Hodge star operators: $*_0, *_1, *_2$, are diagonal matrices.
- Entries $|\star\sigma|/|\sigma|$
- Primal and dual cochain complexes

$$\begin{tikzcd}
C^0(X) \arrow[r, "d_0"] \arrow[d, shift left, "*_0"] & C^1(X) \arrow[r, "d_1"] \arrow[d, shift left, "*_1"] & C^2(X) \arrow[d, shift left, "*_2"] \\
D^2(\star X) \arrow[u, shift left, "*_0^{-1}"] & D^1(\star X) \arrow[l, "-d_0^T"', "d_1^{\mathrm{dual}}"] \arrow[u, shift left, "*_1^{-1}"] & D^0(\star X) \arrow[l, "d_1^T"', "d_0^{\mathrm{dual}}"] \arrow[u, shift left, "*_2^{-1}"]
\end{tikzcd}$$

Hodge-Laplace problem mixed form (matrix version)

Cochains version

For $f \in C^k(X)$ find $(\sigma, u, p) \in C^{k-1}(X) \times C^k(X) \times \mathfrak{H}^k(X)$ such that,

$$\begin{aligned} \langle \sigma, \tau \rangle - \langle u, d\tau \rangle &= 0 && \forall \tau \in C^{k-1}(X), \\ \langle d\sigma, v \rangle + \langle du, dv \rangle + \langle p, v \rangle &= \langle f, v \rangle && \forall v \in C^k(X), \\ \langle u, q \rangle &= 0 && \forall q \in \mathfrak{H}^k(X), \end{aligned}$$

Matrix version

$$\begin{bmatrix} -*_{k-1} & d_{k-1}^T *_k \\ *_k d_{k-1} & d_k^T *_{k+1} d_k \end{bmatrix} \begin{bmatrix} \sigma \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ *_k (f - p) \end{bmatrix}$$

subject to: $Q^T * u = 0$

Non-mixed formulations for DEC

Unknown u primal cochain

$$\Delta_k u = \pm *_k^{-1} d_{n-(k+1)}^{\mathrm{dual}} *_{k+1} d_k u + \pm d_{k-1} *_{k-1}^{-1} d_{n-k}^{\mathrm{dual}} *_k u = f - p$$

Unknown u dual cochain

Dual version of above

Connecting DEC and FEEC

DEC space $C^k(X_h)$ is isomorphic to lowest degree of Whitney forms $\mathcal{P}_1^-\Lambda^k(X_h)$

Definition (DEC and FEEC inner product)

For $\omega, \eta \in \mathcal{P}_1^-\Lambda^k(\Omega)$ the *DEC inner product* $\langle \omega, \eta \rangle_D$ is

$$\langle \omega, \eta \rangle_D := \sum_{\sigma \in X} \frac{|\star\sigma|}{|\sigma|} R(\omega)(\sigma)\, R(\eta)(\sigma)\,,$$

and the *FEEC inner product* is

$$\langle \omega, \eta \rangle_F := \int_\Omega \langle \omega, \eta \rangle \, d\mu\,.$$

$R : \mathcal{P}_1^-\Lambda^k(X) \to C^k(X)$ is the de Rham map $\langle R(\alpha), c \rangle = \int_c \alpha$ for k-chain c and Whitney form α

Norm equivalence roadmap

- First for single acute triangle
- Then for acute triangle mesh
- Then for Delaunay mesh

Norm Equivalence: Acute Triangulations ($k = 1$) (1/5)

Lemma (Single triangle, basis form in $\mathcal{P}_1^-\Lambda^1(T)$)

Let T be a triangle with geometric parameter δ_T, where $\delta_T := \min_{i \neq j \neq k} \frac{|\langle e_i, e_j \rangle|}{\|e_k\|^2}$. For basis form ω_i,

$$c_1(\delta_T)\|\omega_i\|_F^2 \leq \|\omega_i\|_D^2 \leq c_2(\delta_T)\|\omega_i\|_F^2$$

Proof sketch.

- $\|\omega_2\|_F^2 = \frac{c^2 + 3ab\cos C}{12ab\sin C}$.
- $\|\omega_2\|_D^2 = \frac{\cos C}{2\sin C}$.
- Ratio $\frac{\|\omega_2\|_F^2}{\|\omega_2\|_D^2} = \frac{c^2 + 3ab\cos C}{6ab\cos C} \geq \frac{1}{2}$.
- Using $ab\cos C \geq \delta_T c^2$, we get the other bound.

$\square$

Norm Equivalence: Acute Triangulations ($k = 1$) (2/5)

Lemma (Single triangle, any form in $\mathcal{P}_1^- \Lambda^1(T)$, lower bound)

For any $\alpha \in \mathcal{P}_1^- \Lambda^1(T)$ *with* $\delta_T > 0$, $\|\alpha\|_D^2 \geq c_1(\delta_T)\|\alpha\|_F^2$.

Proof sketch.

Let $\alpha = \sum \alpha^i \omega_i$.

$$\|\alpha\|_F^2 \leq 2 \sum \|\alpha^i \omega_i\|_F^2 \leq 2 \left(\frac{1 + 3\delta_T}{6\delta_T} \right) \sum \|\alpha^i \omega_i\|_D^2 = C \|\alpha\|_D^2$$

Diagonal nature of DEC norm for basis forms is key here. $\square$

Norm Equivalence: Acute Triangulations ($k = 1$) (3/5)

Lemma (Single triangle, any form in $\mathcal{P}_1^- \Lambda^1(T)$, upper bound)

For $\alpha \in \mathcal{P}_1^- \Lambda^1(T)$, $\|\alpha\|_D^2 \leq c_2(\delta_T)\|\alpha\|_F^2$.

Proof sketch.

- FEEC norm for 1-forms is scale invariant (Christiansen and Halvorsen 2011)
- DEC norm for 1-forms in 2D is scale invariant (ratio of lengths)
- Scaling argument and compactness of the set of diameter 1 triangles with angles $\leq \pi/2$.
- Other ingredients: shape regularity, some geometry of acute triangles

$\square$

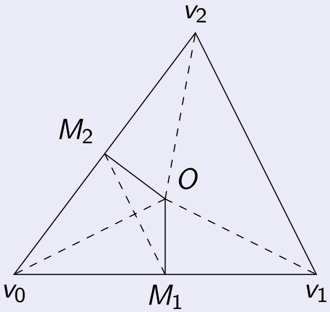
Norm Equivalence: Acute Triangulations ($k = 0$) (4/5)

Lemma

For acute T and $\alpha \in \mathcal{P}_1^- \Lambda^0(T)$, $c_1 \|\alpha\|_F^2 \leq \|\alpha\|_D^2 \leq c_2 \|\alpha\|_F^2$.

Proof sketch.

- For acute triangle, dual area μ_i of vertex v_i satisfies $\mu/4 \leq \mu_i \leq \mu/2$.



- FEEC mass matrix is positive definite

From single triangle to triangulation (5/5)

Theorem (Norm equivalence of DEC norm and FEEC norm)

Given a uniformly acute family $\{X_h\}$ triangulating a piecewise linear surface Ω there exists a constant $c_1(\delta), c_2(\delta) > 0$ independent of h such that

$$c_1(\delta)\, \|\alpha\|_F^2 \leq \|\alpha\|_D^2 \leq c_2(\delta)\, \|\alpha\|_F^2 \,,$$

for all $\alpha \in \mathcal{P}_1^- \Lambda^k(\Omega)$ for $k = 0, 1, 2$.

Proof.

$k = 1$: for single triangle T

$$c_1(\delta_T)\|\alpha|_T\|_F^2 \leq \|\alpha|_T\|_D^2 \leq c_2(\delta_T)\|\alpha|_T\|_F^2 \,,$$

where $c_1(x) = 3x/(1+3x)$ is monotonic increasing and $c_2(x) = 1/(2\sqrt{x})$ is monotonic decreasing. $\square$

continued.

Thus

$$\|\alpha\|_D^2 = \sum_{T\in X_h} \|\alpha|_T\|_D^2 \leq \sum_{T\in X_h} c_2(\delta_T)\|\alpha|_T\|_F^2 = c_2(\delta)\|\alpha\|_F^2\,,$$

and

$$\|\alpha\|_D^2 = \sum_{T\in X_h} \|\alpha|_T\|_D^2 \geq \sum_{T\in X_h} c_1(\delta_T)\|\alpha|_T\|_F^2 = c_1(\delta)\|\alpha\|_F^2\,.$$

- $k=2$: DEC and FEEC norms are the same.
- $k=0$: Constants only depend on acuteness, not on other geometry.

□

Relaxing acuteness constraint

- DEC Hodge star can be defined for non-degenerate Delaunay, but analysis more complicated
- $k = 1$:
  - For (non-degenerate) Delaunay pair, dual edge is positive length but may be negative for single triangle
  - Triangle-by-triangle analysis will not work so entire mesh family is considered
  - Boundary dual can be negative so must require boundary acuteness
- $k = 0$:
  - Dual area need not be bounded by fractions of triangle area
  - Interior vertex and boundary vertex cases have to be considered separately
  - Triangle-by-triangle analysis does not work
  - Star of vertex plays a role

Norm equivalence for Delaunay

Theorem

Let Ω be a piecewise linear triangle mesh surface and $\{X_h\}$ a shape regular, uniformly Delaunay, uniformly boundary acute and curvature bounded family of meshes triangulating Ω. Then for all $\alpha_h \in \mathcal{P}_1^- \Lambda^k(X_h)$ there exist constants c_1, c_2 independent of h such that

$$c_1 \|\alpha_h\|_F^2 \le \|\alpha_h\|_D^2 \le c_2 \|\alpha_h\|_F^2 \,.$$

- With norm equivalence, stability and convergence follow from variational crimes framework of Holst and Stern (2012)
- Our proof however, requires only the right-hand side of this equivalence (the upper bound)
- Thus the uniformly Delaunay condition can be relaxed to a nondegenerate Delaunay condition.

Graph norms

Definition (FEEC and DEC Graph Norm)

For $u_h, v_h \in V_h^k = \mathcal{P}_1^- \Lambda^k(X_h)$ the space V_h^k is endowed with the inner product associated with the graph norm for FEEC Arnold, Falk, and Winther (2010) as

$$\langle u_h, v_h \rangle_{V^k} = \langle u_h, v_h \rangle_F + \langle du_h, dv_h \rangle_F$$

For DEC as

$$\langle u_h, v_h \rangle_{V_h^k}^D := \langle u_h, v_h \rangle_D + \langle du_h, dv_h \rangle_D$$

DEC FEEC consistency error

Theorem (Consistency Error)

Given a family $\{X_h\}$ of DEC-regular triangulations of Ω, let $\alpha_h, \beta_h \in \mathcal{P}_1^- \Lambda^k(X_h)$ for a mesh X_h. Then

$$|\langle \alpha_h, \beta_h \rangle_D - \langle \alpha_h, \beta_h \rangle_F| \leq Ch\left(\|\alpha_h\|_V \|\beta_h\|_V\right).$$

Proof roadmap.

- $k = 0$: α, β both constant, one constant, neither constant
- $k = 1$: both closed (hence exact), one exact, neither exact

□

DEC Poincaré inequality

Theorem (DEC Poincaré inequality)

Given a family $\{X_h\}$ of DEC-regular triangulations of Ω a piecewise linear triangle mesh surface, we have

$$\|v\|_V \leq \hat{c}_P \|dv\|_V \quad \text{and} \quad \|v\|_{V_h}^D \leq \hat{c}_P \|dv\|_{V_h}^D \quad \forall v \in \mathfrak{Z}_h^{\perp_D},$$

where $\hat{c}_P$ is a constant.

Proof outline.

- $\mathfrak{Z}_h^{\perp_D} \neq \mathfrak{Z}_h^{\perp}$. Decompose $v = v^{\perp} + v_0$, where $v^{\perp} \in \mathfrak{Z}_h^{\perp}$ and $v_0 \in \mathfrak{Z}_h$.
- Consistency estimate

$$|\langle v, \alpha\rangle_D - \langle v, \alpha\rangle_F| \leq Ch(\|v\|_F \|d\alpha\|_F + \|dv\|_F \|\alpha\|_F) \qquad \forall \alpha \in V_h\,.$$

- Use $\alpha = v_0$ to get

$$\|v_0\|_F^2 \leq C\, h \|dv^{\perp}\|_F \, \|v_0\|_F$$

Proof outline (contd.)

- Apply FEEC Poincaré inequality to bound $\|v\|_F$ by $\|dv\|_F$.
- $\|v_0\|_V \leq C\,h\,\|dv^\perp\|_F$ and
  $\|v\|_V \leq \|v_0\|_V + \|v^\perp\|_V \leq C\,h\,\|dv^\perp\|_F + c_P\,\|dv^\perp\|_V \leq (C\,h + c_P)\|dv^\perp\|_V$
- Thus $\|v\|_V \leq (C\,h + c_P)\|dv\|_V$
- Result follows for DEC norm using $\|v\|_D \leq c\|v\|_F$.

$\square$

DEC and FEEC harmonic forms

Theorem (DEC and FEEC harmonic forms)

Let $\hat{\mathfrak{H}}_h$ denote the space of DEC harmonic forms, and $\mathfrak{H}_h$ the space of FEEC harmonic forms. Then there exists a bijection $\Pi_h : \hat{\mathfrak{H}}_h \to \mathfrak{H}_h$ such that

$$\|\Pi_h(\hat{p}) - \hat{p}\|_F \leq \hat{C}\, h\, \|\hat{p}\|_F, \qquad \forall\ \hat{p} \in \hat{\mathfrak{H}}_h\,.$$

Stability

Theorem (DEC stability using DEC norm and DEC harmonic forms)

Let $\{X_h\}$ be a family of DEC-regular triangulations of Ω. Then there exists a $\gamma > 0$, depending only on δ_0 and c_P in FEEC Poincaré inequality, such that for any $(\sigma, u, p) \in \mathcal{P}_1^- \Lambda^{k-1}(X_h) \times \mathcal{P}_1^- \Lambda^k(X_h) \times \hat{\mathfrak{H}}_h^k$, when mesh size h is small enough, there exists $(\tau, v, q) \in \mathcal{P}_1^- \Lambda^{k-1}(X_h) \times \mathcal{P}_1^- \Lambda^k(X_h) \times \hat{\mathfrak{H}}_h^k$ with

$$B_D(\sigma, u, p; \tau, v, q) \geq \gamma \left(\|\sigma\|_{V_h}^D + \|u\|_{V_h}^D + \|p\| \right) \left(\|\tau\|_{V_h}^D + \|v\|_{V_h}^D + \|q\| \right).$$

Proof sketch.

- Test functions as in (Arnold, Falk, and Winther 2010)
- Expand bilinear form: $B_D = \|\sigma\|_D^2 - c^{-2}\langle \sigma, \rho \rangle_D + \|d\sigma\|_D^2 + \|du\|_D^2 + \|p\|_D^2 + \ldots$
- Switch to FEEC: Use consistency theorem to replace DEC inner products with FEEC inner products $+O(h)$ error terms.

$\square$

Convergence

Theorem (DEC error using FEEC norm and DEC harmonic forms)

Let $\{X_h\}$ be a family of DEC regular triangulations of Ω. Let $(\hat{u}_h, \hat{\sigma}_h, \hat{p}_h)$ be the DEC solution and (u_h, σ_h, p_h) the FEEC solution. Then we have the error estimate:

$$\|\hat{\sigma}_h - \sigma_h\|_V + \|\hat{u}_h - u_h\|_V + \|\hat{p}_h - p_h\|_F \leq C\,h\left(\|\sigma_h\|_V + \|u_h\|_V + \|p_h\|_F + \|f_h\|_V\right).$$

Summary

- Convergence theory for DEC Hodge-Laplace Problem in 2D.
- Mesh requirements for equivalence between DEC and FEEC norms.
- Consistency between DEC and FEEC inner products for 0, 1, and 2 forms.
- Stability and $O(h)$ convergence rate using FEEC machinery and consistency estimates.
- Bijection and error control between DEC and FEEC harmonic forms.

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Challenges in three dimensions

- On a single tetrahedron equivalence of DEC and FEEC norms for discrete closed forms is not true
- Numerical tests show that for constant forms on good meshes, $\| \cdot \|_D \approx \| \cdot \|_F$.
- Consider the inner product of a constant 1-form (vector in $\mathbb{R}^3$) as a quadratic form defined by a 3×3 matrix.
- While the matrices differ, their traces are identical for DEC and FEEC.