Boundary Dual and Codifferential in DEC

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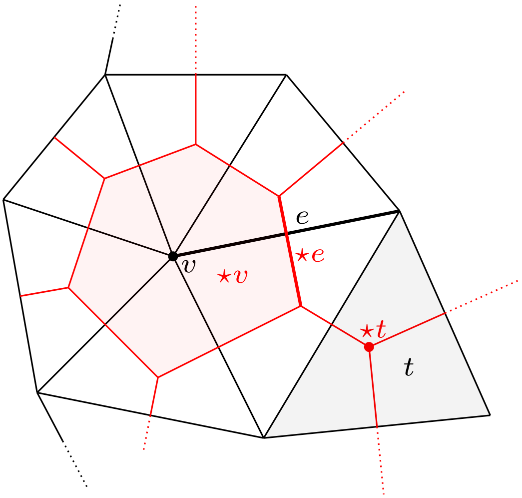
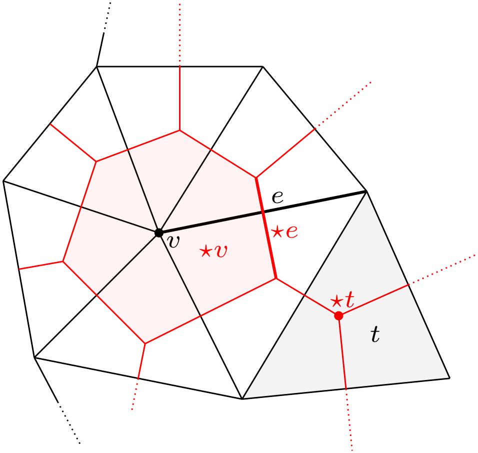
- Based on Chapter 3 of *Discretization of Differential Geometry for Computational Gauge Theory*, PhD thesis of Mark Schubel, 2018
- Similar recent ideas: Chris Eldred (Sandia) and collaborators

- Primal and dual meshes
- Discrete d when no boundary (or homogeneous b.c.)
- Discrete d with boundary (and non-homogeneous b.c.)
- Adjointness of discrete d and δ and the boundary term

Dual mesh in DEC



Primal and Dual Cochains in Dimension 2



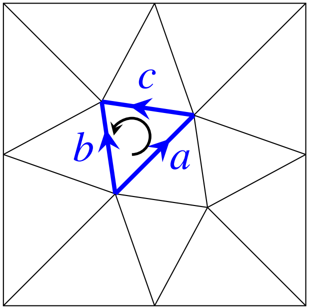
$$
\begin{tikzcd}
C^0(X) \arrow[r, "d_0"] \arrow[d, shift left, "*_0"] & C^1(X) \arrow[r, "d_1"] \arrow[d, shift left, "*_1"] & C^2(X) \arrow[d, shift left, "*_2"] \\
D^2(\star X) \arrow[u, shift left, "*_0^{-1}"] & D^1(\star X) \arrow[l, "d_1^{\mathrm{dual}}"'] \arrow[u, shift left, "*_1^{-1}"] & D^0(\star X) \arrow[l, "d_0^{\mathrm{dual}}"'] \arrow[u, shift left, "*_2^{-1}"]
\end{tikzcd}
$$

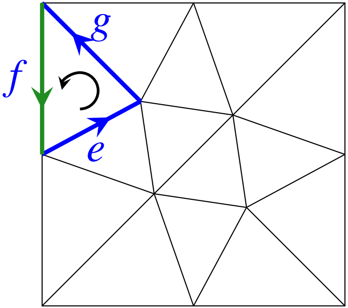
Primal and dual cochains in dimension 3

$$
\begin{tikzcd}
C^0(X) \arrow[r, "d_0"] \arrow[d, shift left, "*_0"] & C^1(X) \arrow[r, "d_1"] \arrow[d, shift left, "*_1"] & C^2(X) \arrow[r, "d_2"] \arrow[d, shift left, "*_2"] & C^3(X) \arrow[d, shift left, "*_3"] \\
D^3(\star X) \arrow[u, shift left, "*_0^{-1}"] & D^2(\star X) \arrow[l, "d_2^{\mathrm{dual}}"'] \arrow[u, shift left, "*_1^{-1}"] & D^1(\star X) \arrow[l, "d_1^{\mathrm{dual}}"'] \arrow[u, shift left, "*_2^{-1}"] & D^0(\star X) \arrow[l, "d_0^{\mathrm{dual}}"'] \arrow[u, shift left, "*_3^{-1}"]
\end{tikzcd}
$$

Primal exterior derivative $\quad d_k : C^k(X) \to C^{k+1}(X)$

Does not need special treatment near boundary





$$a - b + c \qquad\qquad e + f + g$$

Dual exterior derivative d_{n-k}^{dual}

- Mesh dimension $= n$, dual dimension $= n - k$, so primal dimension $= k$
- *Does* need special treament near boundary
- Reason: how dual and primal exterior derivative matrices are related

How d^{dual}_{n-k} and primal d_{k-1} related

When boundary terms are not accounted for in d^{dual}_{n-k}

$$d^{\text{dual}}_{n-k} = (-1)^k \, d^T_{k-1}$$

$$\begin{tikzcd}
C^0(X) \arrow[r, "d_0"] \arrow[d, shift left, "*_0"] & C^1(X) \arrow[r, "d_1"] \arrow[d, shift left, "*_1"] & C^2(X) \arrow[r, "d_2"] \arrow[d, shift left, "*_2"] & C^3(X) \arrow[d, shift left, "*_3"] \\
D^3(\star X) \arrow[u, shift left, "*_0^{-1}"] & D^2(\star X) \arrow[l, "-d_0^T"', "d_2^{\text{dual}}"] \arrow[u, shift left, "*_1^{-1}"] & D^1(\star X) \arrow[l, "d_1^T"', "d_1^{\text{dual}}"] \arrow[u, shift left, "*_2^{-1}"] & D^0(\star X) \arrow[l, "-d_2^T"', "d_0^{\text{dual}}"] \arrow[u, shift left, "*_3^{-1}"]
\end{tikzcd}$$

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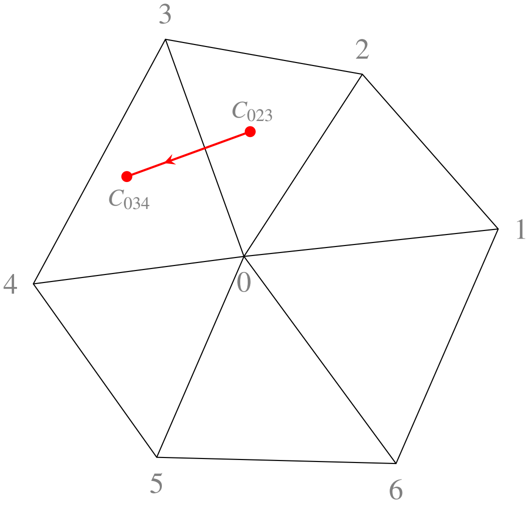
D^2(\star X) \arrow[u, shift left, "*_0^{-1}"] & D^1(\star X) \arrow[l, "-d_0^T"', "d_1^{\text{dual}}"] \arrow[u, shift left, "*_1^{-1}"] & D^0(\star X) \arrow[l, "d_1^T"', "d_0^{\text{dual}}"] \arrow[u, shift left, "*_2^{-1}"]
\end{tikzcd}$$

Dual exterior derivative $\quad d_0^{\text{dual}} : D^0(\star X) \to D^1(\star X)$

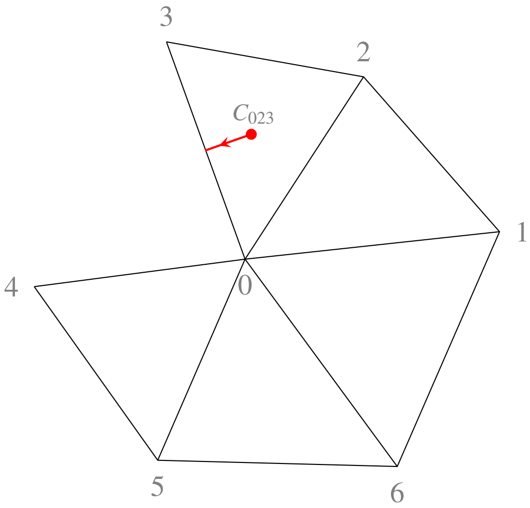
Interior



$$d_0^{\text{dual}} = d_1^T \text{ ``=''} \; \partial_2$$

Dual exterior derivative $d_0^{\text{dual}} : D^0(\star X) \to D^1(\star X)$

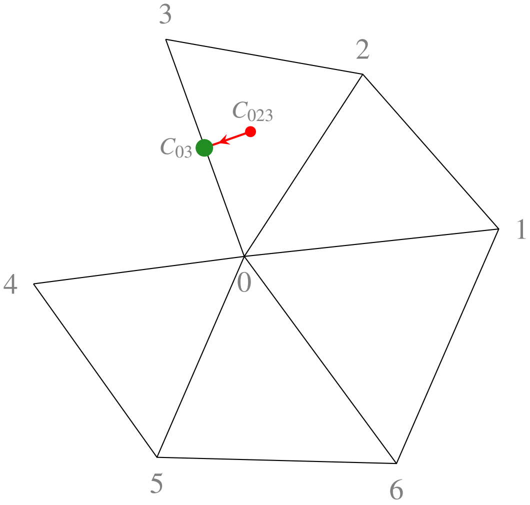
Near boundary, without accounting for boundary terms



$$d_0^{\text{dual}} = d_1^T \text{ ``}{=}\text{'' } \partial_2$$

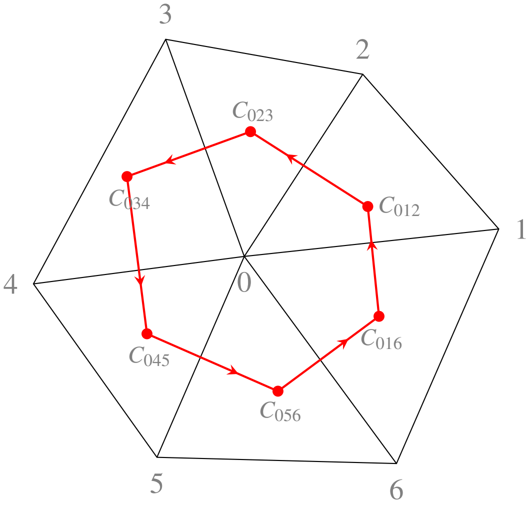
Extended dual exterior derivative $D^0(\star X) \oplus D^0(\star\partial X) \to D^1(\star X)$

Near boundary, with boundary terms incorporated



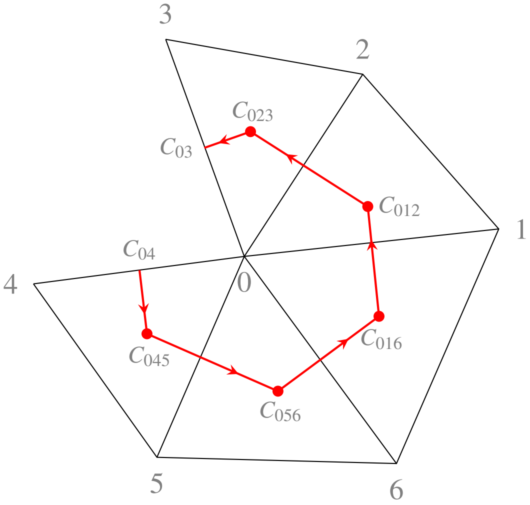
Dual exterior derivative $\quad d_1^{\text{dual}} : D^1(\star X) \to D^2(\star X)$

Interior



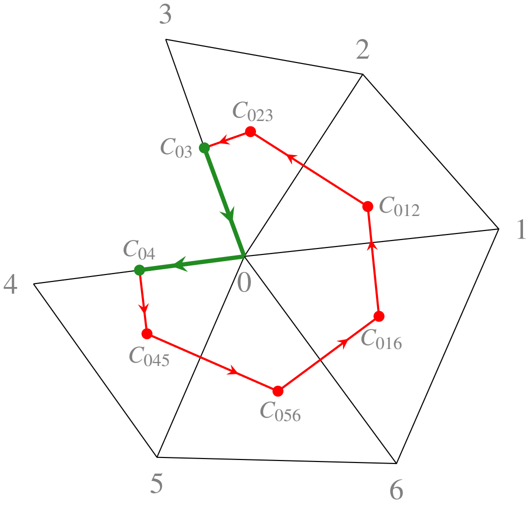
Dual exterior derivative $\quad d_1^{\text{dual}} : D^1(\star X) \to D^2(\star X)$

Near boundary, without accounting for boundary terms

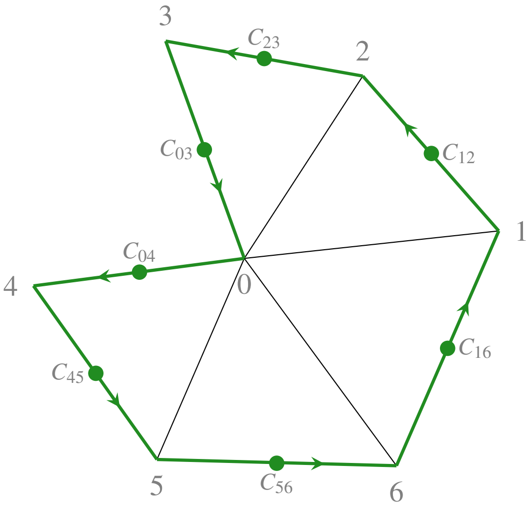


Extended dual exterior derivative $D^1(\star X) \oplus D^1(\star\partial X) \to D^2(\star X)$

Near boundary, with boundary terms incorporated

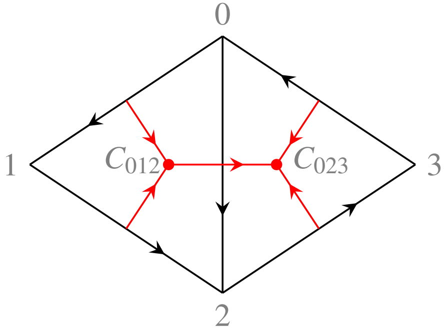


Boundary dual mesh in dimension 2



Dual exterior derivative d_0^{dual}

Without accounting for boundary terms



$$
d_0^{\text{dual}} = d_1^T =
\begin{array}{c|cc|}
 & \star 012 & \star 023 \\
\hline
\star 01 & 1 & 0 \\
\star 02 & -1 & 1 \\
\star 30 & 0 & 1 \\
\star 12 & 1 & 0 \\
\star 23 & 0 & 1 \\
\hline
\end{array}
$$

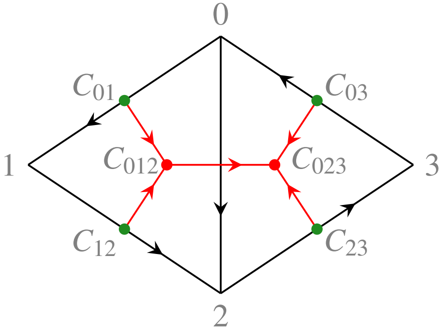
Dual exterior derivative d_0^{dual}

Boundary terms included

$$
d_0^{\text{dual}} = [\ d_1^T,\ -i_1^{\partial}\ ] = \quad
\begin{array}{c}
\\ \star 01 \\ \star 02 \\ \star 30 \\ \star 12 \\ \star 23
\end{array}
\begin{array}{cccccc}
\star 012 & \star 023 & \star\partial 01 & \star\partial 30 & \star\partial 12 & \star\partial 23 \\
\hline
\multicolumn{1}{|c}{1} & \multicolumn{1}{c|}{0} & -1 & 0 & 0 & 0 \\
\multicolumn{1}{|c}{-1} & \multicolumn{1}{c|}{1} & 0 & 0 & 0 & 0 \\
\multicolumn{1}{|c}{0} & \multicolumn{1}{c|}{1} & 0 & -1 & 0 & 0 \\
\multicolumn{1}{|c}{1} & \multicolumn{1}{c|}{0} & 0 & 0 & -1 & 0 \\
\multicolumn{1}{|c}{0} & \multicolumn{1}{c|}{1} & 0 & 0 & 0 & -1 \\
\cline{1-2}
\end{array}
$$

Dual exterior derivative d_0^{dual}

Boundary terms incorporated



	$\star 012$	$\star 023$	$\star\partial 01$	$\star\partial 30$	$\star\partial 12$	$\star\partial 23$
$\star 01$	1	0	−1	0	0	0
$\star 02$	−1	1	0	0	0	0
$\star 30$	0	1	0	−1	0	0
$\star 12$	1	0	0	0	−1	0
$\star 23$	0	1	0	0	0	−1

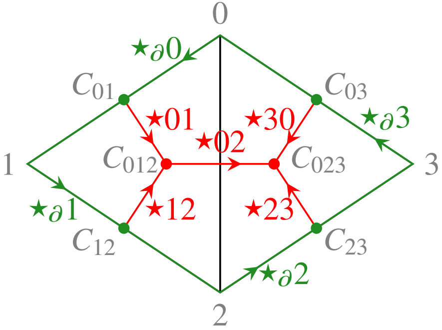
Dual exterior derivative d_1^{dual}

Boundary terms included

$$
d_1^{\mathrm{dual}} = [\, -d_0^T,\ i_0^{\partial}\,] =
\begin{array}{c}
\\
\star 0\\ \star 1\\ \star 2\\ \star 3
\end{array}
\begin{array}{c}
\begin{array}{ccccccccc}
\star 01 & \star 02 & \star 30 & \star 12 & \star 23 & \star\partial 0 & \star\partial 1 & \star\partial 2 & \star\partial 3
\end{array}\\
\begin{array}{|ccccc|cccc}
\hline
1 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0\\
-1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0\\
0 & -1 & 0 & -1 & 1 & 0 & 0 & 1 & 0\\
0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1\\
\hline
\end{array}
\end{array}
$$

Dual exterior derivative d_1^{dual}

Boundary terms incorporated



$$
\begin{array}{c|ccccc|cccc}
 & \star 01 & \star 02 & \star 30 & \star 12 & \star 23 & \star\partial 0 & \star\partial 1 & \star\partial 2 & \star\partial 3 \\
\star 0 & 1 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\
\star 1 & -1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
\star 2 & 0 & -1 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\
\star 3 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1
\end{array}
$$

Dual exterior derivative d_{n-k}^{dual}

General formula accounting for boundary terms

$$d_{n-k}^{\text{dual}} := \left[(-1)^k d_{k-1}^T, \ (-1)^{k-1} \, i_{k-1}^{\partial} \right]$$

Recall: Codifferential

- Codifferential δ is L^2 adjoint of exterior derivative d (modulo boundary term)
- For $\alpha \in \Omega^k(M)$

$$\delta\alpha := (-1)^{n(k-1)+1} * d * \alpha$$

- For $\omega, \eta \in \Omega^k(M)$

$$(\omega, \eta) := \int_M \omega \wedge * \eta$$

  is their L^2 inner product
- For $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^{k-1}(M)$

$$(d\beta, \alpha) - (\beta, \delta\alpha) = \int_{\partial M} \beta \wedge * \alpha$$

- This is how boundary conditions like $\mathrm{tr} * \alpha$ come about

Discrete codifferential

- For $\alpha \in C^k(X)$
$$\delta_k \, \alpha := (-1)^{n(k-1)+1} \, *_{k-1}^{-1} \, d_{n-k}^{\text{dual}} \begin{bmatrix} *_k \alpha \\ \operatorname{tr} * \alpha \end{bmatrix}$$
- Here $\operatorname{tr} * \alpha$ is discretized via integration on the duals of $(k-1)$-simplices on ∂X
- Because $n - k = (n-1) - (k-1)$
- For $\omega, \eta \in C^k(X)$
$$(\omega, \eta) := \omega^T \, *_k \, \eta$$
- For $\omega^*, \eta^* \in D^{n-k}(\star X)$
$$(\omega^*, \eta^*) := \omega^{*T} \, *_k^{-1} \, \eta^*$$

Adjointness

Proposition

The discrete codifferential is the adjoint of the discrete exterior derivative. That is, for $\alpha \in C^k(X)$ and $\beta \in C^{k-1}(X)$

$$(d_{k-1}\beta, \alpha) - (\beta, \delta_k \alpha) = \beta^T i_{k-1}^T \operatorname{tr} *\alpha$$

Can dualize the whole picture

Future work

- Implement combinatorial orientation algorithm for boundary dual
- Test on boundary value problems for $\Delta u = f$ $\qquad (\mathrm{tr} * u,\ \mathrm{tr} * du)$

Acknowledgments

Siqi Jiao for help with the figures