Contraction Wedge Duality and Double Forms

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Outline

1. Contraction, Lie derivative and PDEs
2. Contraction wedge duality and DEC
3. Double forms and Bianchi sums

Recall: Contraction and Lie derivative

Contraction

For $\alpha \in \Omega^k(M)$ and X a vector field, *contraction* of α is $i_X\alpha \in \Omega^{k-1}(M)$ defined as

$$i_X\alpha(\ldots) = \alpha(X, \ldots)$$

Lie derivative

The *Lie derivative* of α along X is

$$\mathcal{L}_X\alpha := \left.\frac{d}{dt}\right|_{t=0} \varphi_t^*\alpha$$

where φ is the flow of X on M.

Cartan Formula

$$\mathcal{L}_X\alpha = di_X\alpha + i_X d\alpha$$

Example PDE applications

- Mass continuity

$$\partial_t \rho + \mathrm{div}(\rho u) = 0 \quad \iff \quad \partial_t(\rho\,\mu) + d\left(\rho\, i_u \mu\right) = 0$$

  $\rho =$ density and $\mu =$ volume form

- Advection

$$\partial_t \alpha + \mathcal{L}_u \alpha = 0$$

- Navier-Stokes $(u \cdot \nabla)u$ term is vector proxy for

$$\mathcal{L}_u u^\flat - \frac{1}{2} d i_u u^\flat$$

An old contraction formula

$$i_X\alpha = (-1)^{k(n-k)} * (*\alpha \wedge X^\flat)$$

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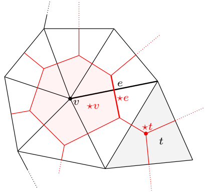
Proof sketch (upto sign):

- Let $e_1, \ldots, e_n$ be orthonormal frame around a point and $e^1, \ldots, e^n$ coframe $[e^i(e_j) = \delta^i_j]$
- Enough to show

$$i_{e_j} e^I = \pm * (*e^I \wedge e^j)$$

  where $e^I = e^{i_1} \wedge \ldots \wedge e^{i_k}$, $i_1 < \ldots < i_k$
- RHS is $\pm * (e^{\bar{I}} \wedge e^j)$ where $\bar{I}$ is complement of I as set
- If $j \notin I$ then LHS=RHS=0
- If $j \in I$ then LHS is $\pm e^{I \setminus \{j\}}$ and RHS is $*(e^{\bar{I}} \wedge e^j) = \pm * e^{\bar{I} \cup j} = \pm e^{I \setminus \{j\}}$

DEC motivation for formula



- Hodge star as ratio of primal and dual

$$\langle \star\alpha, \star\sigma \rangle = \frac{|\star\sigma|}{|\sigma|} \langle \alpha, \sigma \rangle$$

- Wedge product as anti-symmetrized cup

$$\left\langle \alpha \wedge \beta, [v_0, \ldots, v_{k+l}] \right\rangle = \frac{1}{(k+l+1)!} \sum_{\tau \in S_{k+l+1}} \operatorname{sgn}(\tau) \left\langle \alpha \smile \beta, [v_{\tau(0)}, \ldots, v_{\tau(k+l)}] \right\rangle$$

- $i_X \alpha = (-1)^{k(n-k)} * \left(* \alpha \wedge X^\flat \right)$ gives a way to discretize contraction
- Cartan formula $\mathcal{L}_X \alpha = d\, i_X \alpha + i_X d\alpha$ gives a way to discretize Lie derivative

Contraction formula in practice

- Geophysical fluid dynamics:
  - C. Eldred and W. Bauer. "An interpretation of TRiSK-type schemes from a discrete exterior calculus perspective", arXiv:2210.07476.
  - W. Bauer. "A new hierarchically-structured n-dimensional covariant form of rotating equations of geophysical fluid dynamics", Int. J. Geomath, 2016.
- Navier-Stokes:
  - M. S. Mohamed, A. N. Hirani, and R. Samtaney. "Discrete exterior calculus discretization of incompressible Navier-Stokes equations over surface simplicial meshes", J. Comp. Physics, 2016.
  - P. Jagad, A. Abukhwejah, M. Mohamed, and R. Samtaney. "A primitive variable discrete exterior calculus discretization of incompressible Navier-Stokes equations over surface simplicial meshes", Physics of Fluids, 2021.
  - M. Wang, P. Jagad, A. N. Hirani, and R. Samtaney. "Discrete exterior calculus discretization of two-phase incompressible Navier-Stokes equations with a conservative phase field method", J. Comp. Physics, 2023.

Contraction formula as L^2-duality

L^2-duality of contraction and wedge

For k-form α, $(k-1)$-form β

$$(i_X\alpha,\ \beta)_{L^2} = (\alpha,\ X^\flat \wedge \beta)_{L^2}$$

Contraction formula as L^2-duality

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For k-form α, $(k-1)$-form β

$$(X \lrcorner \alpha,\ \beta)_{L^2} = (\alpha,\ X^\flat \wedge \beta)_{L^2}$$

Contraction formula as L^2-duality

L^2-duality of contraction and wedge

For k-form α, $(k-1)$-form β

$$(i_X\alpha,\ \beta)_{L^2} = (\alpha,\ X^\flat \wedge \beta)_{L^2}$$

μ volume form associated with the metric

$$\begin{aligned}
\langle i_X\alpha,\ \beta\rangle\, \mu = \langle \beta,\ i_X\alpha\rangle\, \mu &= \beta \wedge * i_X\alpha \\
&= \beta \wedge (-1)^{k(n-k)} * *(*\alpha \wedge X^\flat) \\
&= (-1)^{k(n-k)+(n-k+1)(k-1)} \beta \wedge (*\alpha \wedge X^\flat) \\
&= (-1)^{k(n-k)+(n-k+1)(k-1)+(n-k)} \beta \wedge (X^\flat \wedge *\alpha) \\
&= (-1)^{(k+1)} ((\beta \wedge X^\flat) \wedge *\alpha) \\
&= (-1)^{(k+1)(k-1)} ((X^\flat \wedge \beta) \wedge *\alpha) \\
&= \langle X^\flat \wedge \beta,\ \alpha\rangle\, \mu = \langle \alpha,\ X^\flat \wedge \beta\rangle\, \mu
\end{aligned}$$

Double forms

q-form valued p-form

$$\Omega^{p,q}(M) = \Omega^p(M; \Lambda^q T^*M) = \Gamma(\Lambda^p T^*M \otimes \Lambda^q T^*M)$$

- Introduced by de Rham and used in differential geometry, analysis, and numerical analysis.
- For $\varphi \in \Omega^{p,q}(M)$ and vector fields $X_1, \ldots, X_p, Y_1, \ldots, Y_q$

$$\varphi(X_1, \ldots, X_p)(Y_1, \ldots, Y_q) \in C^\infty(M)$$

- Decomposable double forms are like $\alpha \otimes \beta$ with $\alpha \in \Omega^p(M)$ and $\beta \in \Omega^q(M)$.

Grid of de Rham sequences for BGG

$$
\begin{tikzcd}
\Omega^{0,0} \arrow[r, "d_\nabla^{00}"] & \Omega^{1,0} \arrow[r, "d_\nabla^{10}"] & \Omega^{2,0} \arrow[r, "d_\nabla^{20}"] & \Omega^{3,0} \arrow[r, "d_\nabla^{30}"] & \Omega^{4,0} \arrow[r] & \cdots \\
\Omega^{0,1} \arrow[r, "d_\nabla^{01}"] \arrow[ur, "s_{01}"] & \Omega^{1,1} \arrow[r, "d_\nabla^{11}"] \arrow[ur, "s_{11}"] & \Omega^{2,1} \arrow[r, "d_\nabla^{21}"] \arrow[ur, "s_{21}"] & \Omega^{3,1} \arrow[r, "d_\nabla^{31}"] \arrow[ur, "s_{31}"] & \Omega^{4,1} \arrow[r] & \cdots \\
\Omega^{0,2} \arrow[r, "d_\nabla^{02}"] \arrow[ur, "s_{02}"] & \Omega^{1,2} \arrow[r, "d_\nabla^{12}"] \arrow[ur, "s_{12}"] & \Omega^{2,2} \arrow[r, "d_\nabla^{22}"] \arrow[ur, "s_{22}"] & \Omega^{3,2} \arrow[r, "d_\nabla^{32}"] \arrow[ur, "s_{32}"] & \Omega^{4,2} \arrow[r] & \cdots \\
\Omega^{0,3} \arrow[r, "d_\nabla^{03}"] \arrow[ur, "s_{03}"] & \Omega^{1,3} \arrow[r, "d_\nabla^{13}"] \arrow[ur, "s_{13}"] & \Omega^{2,3} \arrow[r, "d_\nabla^{23}"] \arrow[ur, "s_{23}"] & \Omega^{3,3} \arrow[r, "d_\nabla^{33}"] \arrow[ur, "s_{33}"] & \Omega^{4,3} \arrow[r] & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{tikzcd}
$$

Bianchi sums s and s^*

$s_{p,q} : \Omega^{p,q}(M) \to \Omega^{p+1,q-1}(M)$

$$(s\varphi)(X_1,\dots,X_{p+1})(Y_1,\dots,Y_{q-1}) = \sum_{i=1}^{p+1}(-1)^{i-1}\varphi(X_1,\dots,\hat{X}_i,\dots,X_{p+1})(X_i,Y_1,\dots,Y_{q-1})$$

$s^*_{p,q} : \Omega^{p,q}(M) \to \Omega^{p-1,q+1}(M)$

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where X's and Y's are vector fields.

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where X's and Y's are vector fields. (Actually only needs vectors at a point.)

Contraction wedge duality and Bianchi sums (1)

Proposition 2.7 in Y. Berchenko-Kogan and E. S. Gawlik. "Finite element spaces of double forms", arXiv:2505.17243:

Alternate characterization of s and s^*

$$s(\alpha \otimes \beta) = \sum_{i=1}^{n} (e^i \wedge \alpha) \otimes (e_i \lrcorner \beta)$$

$$s^*(\alpha \otimes \beta) = \sum_{i=1}^{n} (e_i \lrcorner \alpha) \otimes (e^i \wedge \beta)$$

Proposition 2.8 in arXiv:2505.17243:

Adjointness

s and s^* are adjoints of each other.

- Proof uses contraction wedge duality.

Contraction wedge duality and Bianchi sums (2)

- For $A \in \Omega^{1,1}_{\mathrm{sym}}$ define $i_A : \Omega^{p,q}(M) \to \Omega^{p+1,q-1}(M)$

Contraction wedge duality and Bianchi sums (2)

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- For $g = \sum_{i=1}^{n} e^i \otimes e^i$

$$i_g(\alpha \otimes \beta) = \pm \sum_{i=1}^{n} (e^i \wedge \alpha) \otimes *(e^i \wedge *\beta)$$

Contraction wedge duality and Bianchi sums (2)

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Lemma 3.6 in R. Kupferman and R. Leder. "Elliptic pre-complexes, Hodge-like decompositions and overdetermined boundary-value problems", Forum of Mathematics, Sigma 13, 2025:

s operator and metric

$s = i_g$

- Proof uses contraction wedge duality.

Summary and conclusion

- Contraction is a useful operator in numerical methods for PDEs
- Contraction and wedge are L^2 duals
- This duality shows up in various interpretations of Bianchi sums
- This viewpoint may be useful to construct DEC-like Bianchi sum operators