Discrete Vector Bundles with Connection (and Čech Cohomology)

Daniel Berwick-Evans, Anil N. Hirani, Mark D. Schubel

Department of Mathematics, University of Illinois at Urbana-Champaign

Discretization of Differential Complexes, ESI Vienna, May 21, 2026

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(and perhaps Gemini Pro $20/month for help with tikz?)

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Outline

- Recent history of this field and motivation

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- What structure needs to be replicated by the discrete theory

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- Our construction of discrete vector bundles with connection and various operators

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- How the three constructions are related
- Outlook

Recent history

DISCRETIZATION OF DIFFERENTIAL GEOMETRY FOR COMPUTATIONAL GAUGE THEORY

BY

MARK D SCHUBEL

DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Physics
in the Graduate College of the
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Urbana, Illinois

Doctoral Committee:

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DANIEL BERWICK-EVANS, ANIL N. HIRANI, AND MARK D. SCHUBEL

Abstract. We develop a combinatorial theory of vector bundles with connection on locally ordered simplicial complexes. This is a first step towards a discrete exterior calculus for bundle-valued forms. The basic building block is the discrete exterior covariant derivative, a forward-difference operator defined on bundle-valued cochains. Many standard objects in differential geometry (e.g., curvature, connection 1-forms, gauge transformations) can be understood via the discrete covariant derivative operator, with their defining formulas identical to the smooth setting. These discrete objects satisfy all of the expected algebraic identities, such as naturality with respect to simplicial maps, and a Bianchi identity for discrete curvature. We also show that flat discrete connections determine a cochain complex that computes twisted de Rham cohomology in a local coefficient system determined by the discrete vector bundle, with twisted Poincaré duality (of densities) being one application. Finally, a coarsening operation applied to bundle-valued cochains provides a direct and concrete comparison with the recent framework for discrete bundles of Christiansen and Hu.

1. Introduction

Numerical approaches to partial differential equations often rely on combinatorial models for differential geometric objects. These models can encode structures of interest from a purely mathematical point of view, particularly when the combinatorial model faithfully encodes algebraic identities from geometry. In turn, such algebraic identities ensure that basic geometric concepts are correctly captured by the resulting numerical method.

A fundamental example comes from the algebraic identities relating the gradient, curl, and divergence. These identities can be phrased (and generalized) in terms of the exterior derivative and Hodge star operators acting on differential forms. There are robust combinatorial models for the Hodge-de Rham complex in which the standard algebraic identities hold, e.g., [2, 3, 19, 27, 52]. These combinatorial models and their recent generalizations have met success in a wide variety of applications, e.g., computational electromagnetism [9, 26], elasticity [4, 5, 35], numerical relativity [35, 45], fluid mechanics [28, 30, 38, 42, 43, 50], quantum electrodynamics [44], photonics [39] and other areas of physics, geometry, and computer graphics. The combinatorial wedge product leads to even more sophisticated algebraic structures, e.g., the failure of associativity is encoded by an A_∞-structure [21, 22, 36].

This paper initiates a generalization of these combinatorial models to differential forms with values in a vector bundle. Specifically, we construct combinatorial versions of the exterior covariant derivative d_∇ and various products. In smooth geometry, $d_\nabla \circ d_\nabla$ encodes the curvature of the connection. We define a combinatorial version of curvature via this formula, and then show that the resulting structure satisfies all the expected properties, including a Bianchi identity.

One eventual goal is development of coordinate-independent numerical methods for problems involving bundle-valued forms, e.g., elasticity, Yang–Mills equation (see Remark 1.2), granular defect dynamics in materials and other. These problems can also be studied in combination (such as magnetohydrodynamics in curved spacetime) necessitating a uniform framework that encompasses all the desired applications. These longer-term goals require enhancements of the theory presented below, e.g., a suitable generalization of the combinatorial Hodge star operator of [27].

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2021

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2021, 2023, 2026

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FOUNDATIONS OF COMPUTATIONAL MATHEMATICS

Check for updates

Finite Element Systems for Vector Bundles: Elasticity and Curvature

Snorre H. Christiansen[1] · Kaibo Hu[2]

Received: 1 April 2020 / Revised: 12 October 2021 / Accepted: 12 October 2021

Abstract
We develop a theory of finite element systems, for the purpose of discretizing sections of vector bundles, in particular those arising in the theory of elasticity. In the presence of curvature, we prove a discrete Bianchi identity. In the flat case, we prove a de Rham theorem on cohomology groups. We check that some known mixed finite elements for the stress–displacement formulation of elasticity fit our framework. We also define, in dimension two, the first conforming finite element spaces of metrics with good linearized curvature, corresponding to strain tensors with Saint-Venant compatibility conditions. Cochains with coefficients in rigid motions are given a key role in relating continuous and discrete elasticity complexes.

Keywords Finite elements · Elasticity · Bianchi identity · de Rham theorem

Mathematics Subject Classification 65N30 · 58A10 · 74B05

Introduction

In this paper, we first generalize the previously introduced framework of finite element systems (FES) [26,35] so that it can treat, in particular, elasticity problems, and then

Communicated by Hans Munthe – Kaas.

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A Discrete Exterior Calculus of Bundle-valued Forms

Theo Braune*
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Michigan State U.

François Gay-Balmaz‡
Nanyang Technological University

Mathieu Desbrun§
Inria, Ecole Polytechnique

May 23, 2025

Abstract

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Keywords: Discrete exterior calculus, differential forms, vector bundles, structure-preserving discretization

Mathematical Subject Classification: 53A70

1 Introduction

Over the past few decades, structure-preserving finite-dimensional approximations of Cartan's exterior calculus such as Finite-Element Exterior Calculus (FEEC [2]) or Discrete Exterior Calculus (DEC [15]) have proven effective in fields as varied as computational physics and geometry processing [12]. At their heart is a discretization of smooth, *scalar-valued* differential

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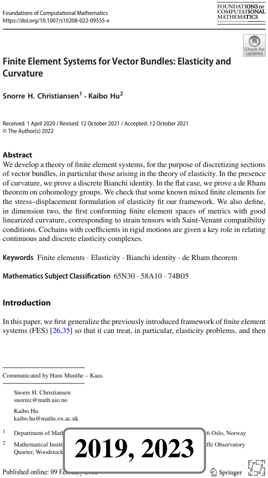
Recent history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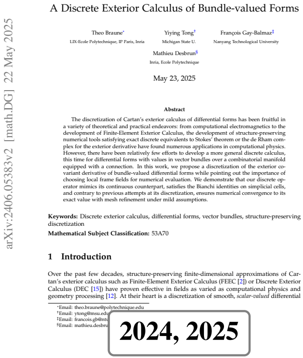
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A Discrete Exterior Calculus of Bundle-valued Forms

Theo Braune* — LIX-Ecole Polytechnique, IP Paris, Inria
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INSTITUT POLYTECHNIQUE DE PARIS

NNT : 20XXIPPAXXXX

Thèse de doctorat

ÉCOLE POLYTECHNIQUE

IP PARIS

Geometry Driven Discretization of Differential Operators

Thèse de doctorat de l'Institut Polytechnique de Paris préparée à Ecole Polytechnique

École doctorale n°636 École doctorale de l'Institut Polytechnique de Paris (EDIPP)
Spécialité de doctorat : Mathématiques et informatique

Thèse présentée et soutenue à Palaiseau, le 28 Avril 2026, par

THEO FRIEDRICH BRAUNE

Composition du Jury :

Anil Hirani Professor, University of Illinois at Urbana-Champaign	Rapporteur
Jacques-Olivier Lachaud Professor, Université de Savoie Mont Blanc	Rapporteur
Francesca Rapetti Professor, Université de Cote d'Azur	Examinateur
David Coeurjolly Directeur de Recherche, CNRS, LIRIS	Examinateur
Jean-Marc Thiery Senior Research Scientist, Adobe Research	Examinateur
Mathieu Desbrun Directeur de Recherche, Inria	Directeur de thèse

2026

Motivation?

$$
\begin{tikzcd}
0 \arrow[r] & \Omega^{0,0} \arrow[r, "d_\nabla"] & \Omega^{1,0} \arrow[r, "d_\nabla"] & \Omega^{2,0} \arrow[r, "d_\nabla"] & \Omega^{3,0} \arrow[r, "d_\nabla"] & \Omega^{4,0} \arrow[r] & \cdots & \Omega^{n,0} \arrow[r] & 0 \\
0 \arrow[r] \arrow[ur] & \Omega^{0,1} \arrow[r, "d_\nabla"] \arrow[ur, "s"'] & \Omega^{1,1} \arrow[r, "d_\nabla"] \arrow[ur, "s"'] & \Omega^{2,1} \arrow[r, "d_\nabla"] \arrow[ur, "s"'] & \Omega^{3,1} \arrow[r, "d_\nabla"] \arrow[ur, "s"'] & \Omega^{4,1} \arrow[r] & \cdots & \Omega^{n,1} \arrow[r] \arrow[ur] & 0 \\
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 & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \\
0 \arrow[r] \arrow[uur] & \Omega^{0,n} \arrow[r, "d_\nabla"] \arrow[uur, "s"'] & \Omega^{1,n} \arrow[r, "d_\nabla"] \arrow[uur, "s"'] & \Omega^{2,n} \arrow[r, "d_\nabla"] \arrow[uur, "s"'] & \Omega^{3,n} \arrow[r, "d_\nabla"] \arrow[uur, "s"'] & \Omega^{4,n} \arrow[r] & \cdots & \Omega^{n,n} \arrow[r] \arrow[uur] & 0
\end{tikzcd}
$$

One row of double forms

- Each node in the diagram is

$$\Omega^{p,q}(M) := \Omega^p(M; \Lambda^q T^*M) = \Gamma(\Lambda^p T^*M \otimes \Lambda^q T^*M)$$

- Each row is a copy of the following for $q = 0, 1, \ldots, n$

$$0 \longrightarrow \Omega^{0,q}(M) \xrightarrow{\nabla^q = d^0_{\nabla^q}} \Omega^{1,q}(M) \xrightarrow{d^1_{\nabla^q}} \Omega^{2,q}(M) \xrightarrow{d^2_{\nabla^q}} \cdots$$

- d_{∇^q} extends ∇^q of $\Lambda^q T^*M$
- ∇^q itself is induced from ∇ on TM

E valued forms

- More generally, replace $\Lambda^q T^*M$ by E

$$\Omega^p(M;E) := \Gamma(\Lambda^p T^*M \otimes E)$$

- Now d_{∇^E} is induced from ∇^E of E over M.
- The sequence becomes

$$0 \longrightarrow \Omega^0(M;E) \xrightarrow{\nabla^E = d^0_{\nabla^E}} \Omega^1(M;E) \xrightarrow{d^1_{\nabla^E}} \Omega^2(M;E) \xrightarrow{d^2_{\nabla^E}} \Omega^3(M;E) \xrightarrow{d_{\nabla^E}} \cdots$$

A note on products

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  For $B = \beta \otimes \Phi \in \Omega^k(M;\mathrm{End}(E))$ and $\alpha = \eta \otimes s \in \Omega^l(M;E)$

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- Endomorphism and endomorphism forms (composition product):
  For $A = \alpha \otimes \Phi \in \Omega^k(M; \mathrm{End}(E))$ and $B = \beta \otimes \Psi \in \Omega^l(M; \mathrm{End}(E))$

$$A \wedge B = (\alpha \wedge \beta) \otimes (\Phi \circ \Psi)$$

Operators and structures to be replicated

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- Exterior covariant derivative $d_\nabla : \Omega^k(M;E) \longrightarrow \Omega^{k+1}(M;E)$
- Product on $\Omega^k(M;E) \times \Omega^l(M)$
- First Leibniz rule:
  For $\alpha \in \Omega^k(M;E)$ and $w \in \Omega^l(M)$:

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- Curvature of the connection:

$$F_\nabla := d_\nabla \circ d_\nabla \in \Omega^2(M;\mathrm{End}(E))$$

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- Product on $\Omega^k(M; \mathrm{End}(E)) \times \Omega^l(M; E)$
- Exterior covariant derivative for endomorphism valued forms

$$d_{\nabla^{\mathrm{End}}} : \Omega^k(M; \mathrm{End}(E)) \longrightarrow \Omega^{k+1}(M; \mathrm{End}(E))$$

- Defined by enforcing a second Leibniz rule:
  For $B \in \Omega^k(M; \mathrm{End}(E))$ and $\alpha \in \Omega^l(M; E)$

$$d_\nabla(B \wedge \alpha) = (d_{\nabla^{\mathrm{End}}} B) \wedge \alpha + (-1)^k B \wedge d_\nabla \alpha$$

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- Product on $\Omega^k(M; \mathrm{End}(E)) \times \Omega^l(M; \mathrm{End}(E))$ and corresponding Leibniz rule

Operators and structure to be replicated

- Bianchi identities:
  - $d_{\nabla^{\mathrm{End}}} F_{\nabla} = 0$
  - For $\alpha \in \Omega^k(M; E)$: $d_{\nabla}(d_{\nabla}\alpha) = F_{\nabla} \wedge \alpha$

Operators and structure to be replicated

- Bianchi identities:
  - $d_{\nabla^{\mathrm{End}}} F_\nabla = 0$
  - For $\alpha \in \Omega^k(M;E)$: $d_\nabla(d_\nabla \alpha) = F_\nabla \wedge \alpha$
- Naturality:
  - A smooth map $f : N \longrightarrow M$ determines a pullback bundle $f^*E \longrightarrow N$ and pullback connection $f^*\nabla$ on f^*E
  - Exterior covariant derivative commutes with pullback
  - Curvature of ∇ on E pulls back to the curvature of $f^*\nabla$ on f^*E

Operators and structure to be replicated

Local structure (more common in physics)

- Assume trivialization of E over an open subset $U \subset M$.
- For a chosen isomorphism of vector bundles $E|_U \simeq U \times \mathbb{R}^r$:

$$d_\nabla \overset{\text{locally}}{=} d + A, \quad F|_U \overset{\text{locally}}{=} F_A := dA + A \wedge A, \qquad A \in \Omega^1(U; \mathrm{End}(\mathbb{R}^r))$$

- A is connection 1-form
- Change of local trivialization of $E|_U$ by $g\colon U \to \mathrm{GL}_r(\mathbb{R})$ (gauge transformation):
  - $A \mapsto gAg^{-1} - dgg^{-1}$
  - $F_A \mapsto gF_Ag^{-1}$

Overview of results

Structure preservation

Theorem

Let X and Y be locally ordered simplicial complexes, E a discrete vector bundle with connection over X, and $f : Y \to X$ an order preserving simplicial map. We construct operations

$$\begin{aligned}
d_\nabla &: C^k(X;E) \to C^{k+1}(X;E) \\
\smile &: C^k(X;E) \times C^l(X) \to C^{k+l}(X;E) \\
\smile &: C^k(X;\mathrm{Hom}(E)) \times C^l(X;E) \to C^{k+l}(X;E) \\
\smile &: C^k(X;\mathrm{Hom}(E)) \times C^l(X;\mathrm{Hom}(E)) \to C^{k+l}(X;\mathrm{Hom}(E)) \\
d_{\nabla^{\mathrm{Hom}}} &: C^k(X;\mathrm{Hom}(E)) \to C^{k+1}(X;\mathrm{Hom}(E)) \\
F &:= d_\nabla \circ d_\nabla \in C^2(X;\mathrm{Hom}(E)) \\
f^* &: C^k(X;E) \to C^k(Y;f^*E)
\end{aligned}$$

Overview of results

Theorem (contd.)

such that for $\alpha \in C^\bullet(X;E)$, $w \in C^\bullet(X)$, $B, D \in C^\bullet(X;\mathrm{Hom}(E))$,

(i) Leibniz rules hold (second one by construction):

$$d_\nabla(\alpha \smile w) = d_\nabla\alpha \smile w + (-1)^{|\alpha|}\alpha \smile dw$$
$$d_\nabla(B \smile \alpha) = (d_{\nabla^{\mathrm{Hom}}}B) \smile \alpha + (-1)^{|B|}B \smile d_\nabla\alpha$$

(ii) the operations are natural:

$$f^*(\alpha \smile w) = f^*\alpha \smile f^*w, \qquad f^*d_\nabla = d_{f^*\nabla}f^*,$$

(iii) square of d_∇ is curvature: $d_\nabla d_\nabla\alpha = F_\nabla \smile \alpha$

(iv) differential Bianchi identity is satisfied: $d_{\nabla^{\mathrm{Hom}}} F_\nabla = 0$

Overview of results

Local structure preservation

Theorem

The discrete 1-form $A := d_\nabla - d$ satisfies

1. the covariant derivative is determined by A as $d_\nabla = d + A$;
2. the connection 1-form A pulls back to a connection 1-form f^*A;
3. the curvature satisfies $F = dA + A \smile A$ and the Bianchi identity $(d + A)F = 0$ holds;
4. under a gauge transformation g of the discrete vector bundle with connection, A transforms as $A \mapsto gAg^{-1} - dgg^{-1}$, and the curvature transforms as $F \mapsto gFg^{-1}$.

Overview of results

Curvature obstruction to anti-symmetrization

- DEC discrete wedge product is anti-symmetrized cup product
- Anti-symmetrizing cup product breaks Leibniz rule

Other results

Not discussed today

For flat connections $d_\nabla^2 = 0$ so can talk about (E-twisted) cohomology

- Results on twisted cohomology and Čech cohomology
- Consequence for twisted Poincaré duality and densities

Important caveats

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  - Parallel transports on edges are given
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- We assume the data is given:
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  - Construction of parallel transport maps requires care
  - Stokes theorem is not true in general
- See the talk of Theo Braune for a solution of these issues
- Our goal was to build the framework assuming these are given

Background

Ingredients

- Locally ordered simplicial complexes category OSiCo
- Partial order on vertices, every simplex totally ordered
- Morphisms are order preserving simplicial maps

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- Locally ordered simplicial complexes category OSiCo
- Partial order on vertices, every simplex totally ordered
- Morphisms are order preserving simplicial maps
- How to construct OSiCo?
  - Define a global total order on vertices?
  - Problem: this is not functorial, simplicial maps may not become order-preserving
- One solution: use subdivision

Subdivision functor

- Subdivision is a functor $\mathsf{SiCo} \to \mathsf{OSiCo}$
- If X (unordered) simplicial complex then $\operatorname{sd} X$ is locally ordered
- If $f : X \to Y$ simplicial map then $\operatorname{sd} f : \operatorname{sd} X \to \operatorname{sd} Y$ is order preserving

Subdivision example

- Simplicial complex $X = \{\{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$

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Subdivision example

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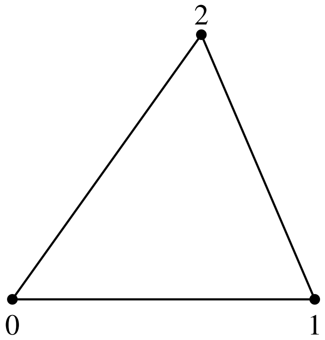
- $\operatorname{sd} X$ vertex set: $\{\{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$
- Six maximal simplices of dimension 2, e.g., $\{\{0, 1, 2\}, \{1, 2\}, \{1\}\}$

Subdivision example (geometric realization)



X

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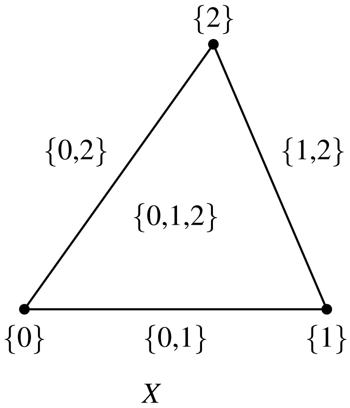

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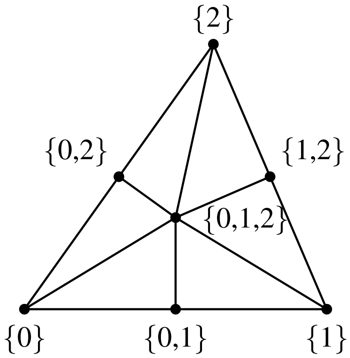

X and sd X

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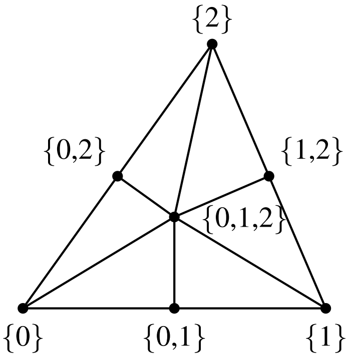

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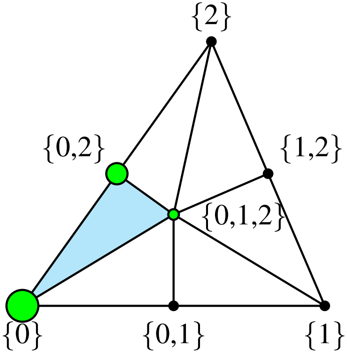
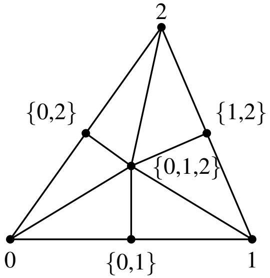


X and sd X

- Vertices in sd X correspond to simplices of X
- Vertices of $\sigma \in \mathrm{sd}\, X$ are from different dimensions of X
- We define the local order as reverse order of dimension

Subdivision of simplicial map



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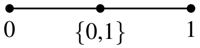

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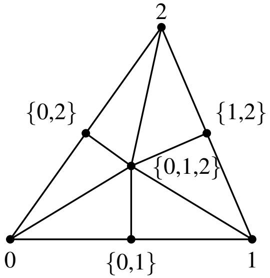

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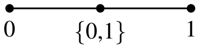
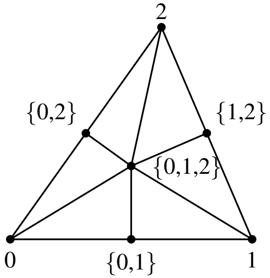


Y and $\operatorname{sd} Y$

- Simplicial map $f : X \to Y$, vertex mapping: $0 \mapsto 0$, $1 \mapsto 1$, $2 \mapsto 0$

Subdivision of simplicial map



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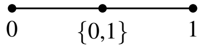

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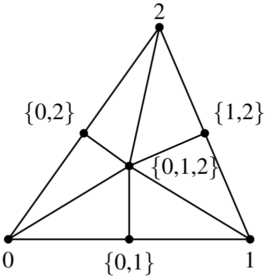

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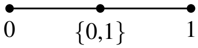

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- sd f : sd $X \to$ sd Y vertex mapping:

$$\begin{array}{lll} \{0\} \mapsto \{0\} & \{0,1\} \mapsto \{0,1\} & \{0,1,2\} \mapsto \{0,1\} \\ \{1\} \mapsto \{1\} & \{0,2\} \mapsto \{0\} & \\ \{2\} \mapsto \{0\} & \{1,2\} \mapsto \{0,1\} & \end{array}$$

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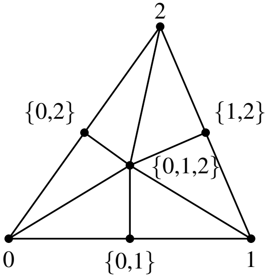
- The cubical cell complex $Q(X)$ is formed by these combinatorial cubes for $0 \leq k, l, k+l \leq n$
- In DEC, for a manifold complex of dimension n and $c(\sigma)$ the circumcenter, $D(\sigma^k \preceq \sigma^n)$ is the dual cell $\star\sigma^k$

Smallest vertex



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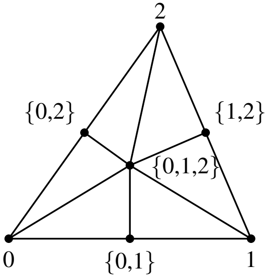

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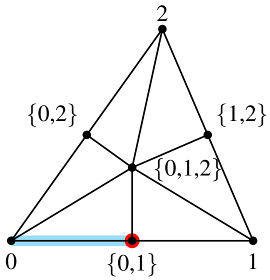

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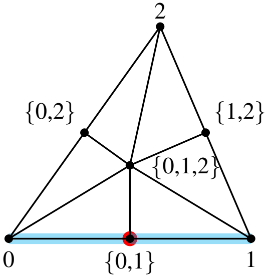

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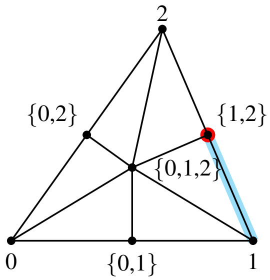

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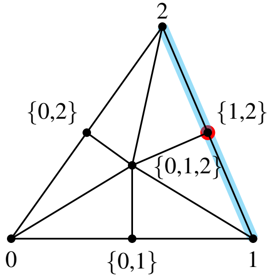

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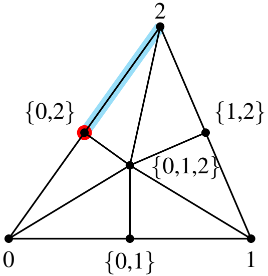

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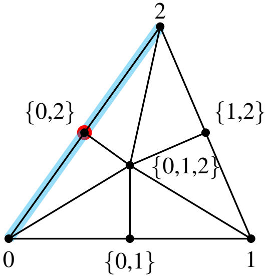

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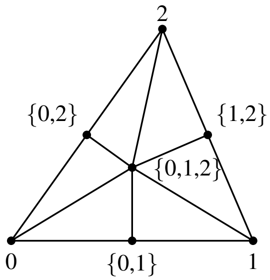

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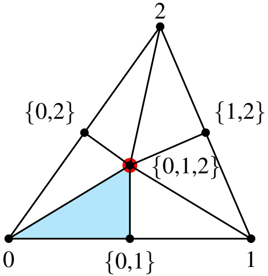

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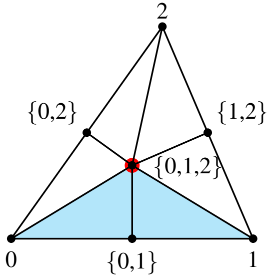

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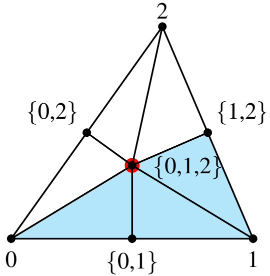

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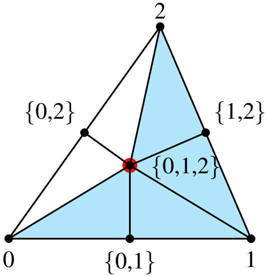

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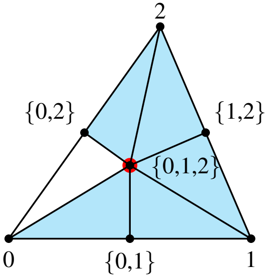

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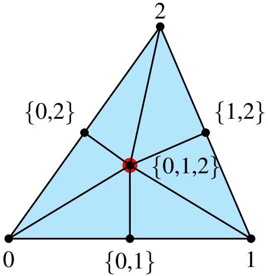

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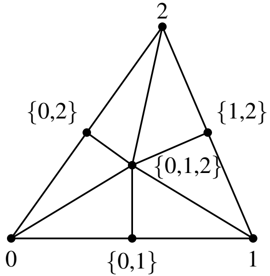

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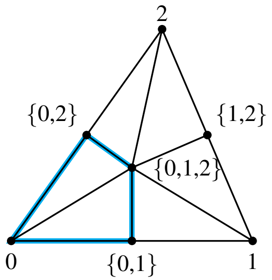

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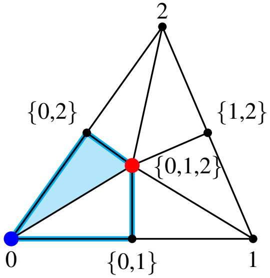

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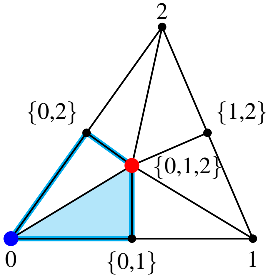

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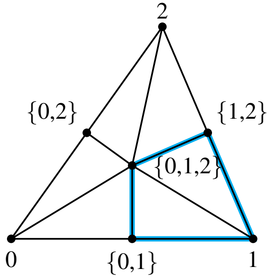

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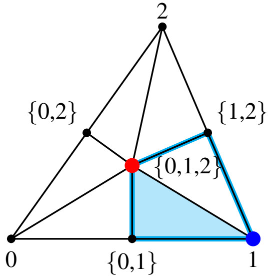

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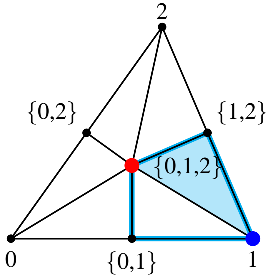

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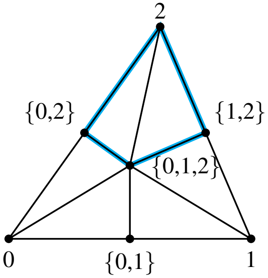

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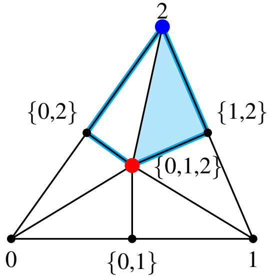

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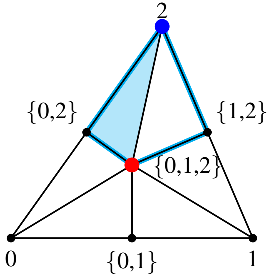

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Derivative, Product, Naturality

Discrete vector bundle with connection

Definition

For a locally ordered simplicial complex X, $E \longrightarrow X$ a real discrete vector bundle with connection over X is data:

1. for each vertex i, a finite-dimensional real vector space E_i (fiber at i)
2. for each edge $[i, j]$, invertible linear map $U_{ji} \colon E_i \longrightarrow E_j$ (parallel transport from i to j).

Compatibility condition $U_{ij} = U_{ji}^{-1}$.

Vector bundle valued discrete forms

Definition

- A vector bundle valued discrete k-form α assigns vector $\left\langle \alpha,\ \sigma^k \right\rangle_l \in E_l$ to σ^k
  (l is smallest vertex of σ^k)
- For $\pi \in S_{k+1}$ a permutation, $\sigma = [v_0, \ldots, v_k]$,
  $\langle \alpha,\ [v_{\pi(0)}, \ldots, v_{\pi(k)}]\rangle_l := \mathrm{sgn}(\pi)\langle \alpha,\ \sigma\rangle_l$ where $\mathrm{sgn}(\pi)$ is the sign of π.
- $C^k(X; E)$ is the vector space of vector bundle valued discrete k-forms
- A section s is a vector bundle valued discrete 0-form
  (map that assigns vector $s_i \in E_i$ for each vertex i)

Discrete covariant derivative

Definition

The discrete covariant derivative $\nabla\colon C^0(X;E) \to C^1(X;E)$ is characterized by

$$\big\langle \nabla s,\ [i,j] \big\rangle_i := U_{ij} s_j - s_i \in E_i, \qquad i < j, \quad s \in C^0(X;E) = \Gamma(E).$$

Pullback bundle

Definition

Given $(E, \nabla) \to X$ and $f : Y \to X$ an order-preserving simplicial map, the pullback bundle over Y is denoted $f^*(E, \nabla) = (f^*E, f^*\nabla)$ and defined as follows. Take as fibers $(f^*E)_i = E_{f(i)}$ for each $i \in Y_0$, and the connection $f^*\nabla$ is defined by assigning to each edge $[i, j] \in Y_1$ the parallel transport map:

$$U_{ij} = \begin{cases} U_{f(i)f(j)} & \text{if } [f(i), f(j)] \in X_1 \\ \mathrm{Id}_{E_{f(i)}} = \mathrm{Id}_{E_{f(j)}} & \text{otherwise.} \end{cases}$$

Pullback of discrete forms

Definition

Let Y and X be locally ordered simplicial complexes, $f : Y \to X$ an order-preserving abstract simplicial map, and $\alpha \in C^k(X; E)$. Then the pullback of α, denoted $f^*\alpha$, is the f^*E-valued discrete form defined by:

$$\langle f^*\alpha,\ [0 \dots k] \rangle_0 := \begin{cases} \big\langle \alpha, f([0 \dots k]) \big\rangle_{f(0)} & \text{if } f([0 \dots k]) \text{ is a } k\text{-simplex in } Y. \\ 0 & \text{otherwise}. \end{cases}$$

Discrete exterior covariant derivative

- Recall how the covariant derivative extends to d_∇ in smooth setting: for $\varphi \in \Omega^p(M;E)$

$$(d_{\nabla^E}\varphi)(X_0,\dots,X_p) = \sum_{0\le i\le p}(-1)^i\nabla^E_{X_i}(\varphi(X_0,\dots,\hat{X}_i\dots X_p)) + \sum_{0\le l<m\le p}(-1)^{l+m}\varphi([X_l,X_m],X_0\dots\hat{X}_l\dots\hat{X}_m\dots X_p) \quad (1)$$

OR: for $\alpha \in \Omega^p(M)$ and $s \in \Gamma(E)$

$$d^p_{\nabla^E}(\alpha\otimes s) = d\alpha\otimes s + (-1)^p\,\alpha\wedge\nabla^E s$$

- We take a different approach

Discrete exterior covariant derivative

For $\alpha \in C^{k-1}(X;E)$ the discrete exterior covariant derivative of α is $d_\nabla \alpha \in C^k(X;E)$ characterized by

$$\big\langle d_\nabla \alpha,\ [0 \dots k]\big\rangle_0 := U_{01}\big\langle \alpha,\ [1 \dots k]\big\rangle_1 + \sum_{i=1}^{k} (-1)^i \big\langle \alpha,\ [0 \dots \hat{\imath} \dots k]\big\rangle_0 .$$

(Inspired by A. Kock [1996])

Leibniz rule

- Does this d_∇ satisfy a Leibniz rule?

Leibniz rule

- Does this d_∇ satisfy a Leibniz rule?
- For that we need a product

Leibniz rule

- Does this d_∇ satisfy a Leibniz rule?
- For that we need a product
- Recall how wedge product was defined in DEC

Recall DEC wedge product

Definition

For $\alpha \in C^k(X)$ and $\beta \in C^l(X)$ the discrete DEC wedge product $\alpha \wedge \beta$ is characterized by

$$\left\langle \alpha \wedge \beta,\ [0 \dots k+l] \right\rangle = \frac{1}{(k+l+1)!} \sum_{\tau \in S_{k+l+1}} \operatorname{sgn}(\tau) \left\langle \alpha \smile \beta,\ [\tau(0) \dots \tau(k+l)] \right\rangle$$

Proposition

Let X be a manifold PL simplicial complex, $\alpha \in C^k(X)$, $\beta \in C^l(X)$ and σ a $(k+l)$-dimensional simplex in X. Then

$$\langle \alpha \wedge \beta,\ \sigma \rangle = \int_\sigma W\alpha \wedge W\beta\,.$$

Product between bundle valued and scalar discrete forms

Due to the curvature obstruction result shown earlier, we do not anti-symmetrize the cup.

Definition

Given $\alpha \in C^k(X;E)$ and $w \in C^l(X)$ their cup product $\alpha \smile w \in C^{k+l}(X;E)$ is defined by its evaluation on a $(k+l)$-simplex at the origin vertex as

$$\langle \alpha \smile w,\ [0 \ldots k+l] \rangle_0 := \langle \alpha,\ [0 \ldots k] \rangle_0 \, \langle w,\ [k \ldots k+l] \rangle$$

Proposition

Given X, Y locally ordered simplicial complexes, bundle $(E, \nabla) \to Y$, and order preserving simplicial map $f : X \to Y$, for any $\alpha \in C^k(Y;E)$ and $w \in C^l(Y)$ and simplex $[0 \ldots k+l]$ in X

$$\langle f^*(\alpha \smile w),\ [0 \ldots k+l] \rangle_0 = \langle f^*\alpha \smile f^*w,\ [0 \ldots k+l] \rangle_0$$

First Leibniz rule

Proposition

For $\alpha \in C^k(X;E)$ and $w \in C^l(X)$, the discrete covariant derivative d_∇ satisfies the following Leibniz rule

$$\langle d_\nabla(\alpha \smile w),\ [0\dots k{+}l{+}1]\rangle_0 = \langle d_\nabla \alpha \smile w,\ [0\dots k{+}l{+}1]\rangle_0 + (-1)^k \langle \alpha \smile dw,\ [0\dots k{+}l{+}1]\rangle_0.$$

Curvature

Homomorphism valued discrete forms

Definition

A homomorphism-valued discrete k-form assigns to each k-simplex $[0 \dots k]$ a linear map $E_k \longrightarrow E_0$ from the highest to the lowest vertex. The space of homomorphism-valued discrete k-forms is denoted $C^k(X; \mathrm{Hom}(E))$. Given $B \in C^k(X; \mathrm{Hom}(E))$ and $\alpha \in C^l(X; E)$ the action of B on α is $B \smile \alpha$ defined by:

$$\langle B \smile \alpha,\ [0 \dots k+l] \rangle_0 := \langle B,\ [0 \dots k] \rangle_{0,k} \langle \alpha,\ [k \dots k+l] \rangle_k \ .$$

- The subscript in $\langle B,\ [0 \dots k] \rangle_{0,k}$ indicates that the value is a map $E_k \longrightarrow E_0$.

Discrete curvature

Proposition

$$d_\nabla^2 = d_\nabla \circ d_\nabla \in C^2(X;\mathrm{Hom}(E))$$

Definition

The discrete curvature is defined as $F_\nabla := d_\nabla^2 \in C^2(X;\mathrm{Hom}(E))$. Its value on a triangle is

$$\langle F_\nabla,\ [0,1,2]\rangle_{0,2} = U_{01}U_{12} - U_{02}$$

Exterior covariant derivative for homomorphism valued discrete forms

Definition

The discrete exterior covariant derivative on homomorphism-valued discrete k-forms $d_{\nabla^{\mathrm{Hom}}} : C^k(X; \mathrm{Hom}(E)) \to C^{k+1}(X; \mathrm{Hom}(E))$ is characterized by demanding:

$$d_\nabla (B \smile \alpha) = (d_{\nabla^{\mathrm{Hom}}} B) \smile \alpha + (-1)^k B \smile d_\nabla \alpha \,,$$

for $B \in C^k(X; \mathrm{Hom}(E))$ and $\alpha \in C^l(X; E)$.

Exterior covariant derivative for homomorphism valued discrete forms

Proposition

$d_{\nabla^{\mathrm{Hom}}} B \in C^{k+1}(X; \mathrm{Hom}(E))$ evaluates to

$$
\begin{aligned}
\langle d_{\nabla^{\mathrm{Hom}}} B,\ [0 \ldots k+1] \rangle_{0,k+1} = & U_{01} \langle B,\ [1 \ldots k+1] \rangle_{1,k+1} \\
& + \sum_{i=1}^{k} \left[(-1)^i \langle B,\ [0 \ldots \hat{i} \ldots k+1] \rangle_{0,k+1} \right] \\
& + (-1)^{k+1} \langle B,\ [0 \ldots k] \rangle_{0,k}\, U_{k,(k+1)} \,.
\end{aligned}
$$

d_∇^2, curvature, and the differential Bianchi identity

Proposition

For a vector bundle-valued discrete k-form $\alpha \in C^k(X; E)$, we have

$$d_\nabla(d_\nabla(\alpha)) = F_\nabla \smile \alpha.$$

Proposition

The discrete curvature satisfies the differential Bianchi identity

$$d_{\nabla^{\mathrm{Hom}}} F_\nabla = 0\,.$$

Connection 1-cochains and gauge transformations

Lemma

Let ∇ and ∇' be two (different) discrete connections on a discrete vector bundle $E \to X$. Then the difference

$$\nabla - \nabla' \in C^1(X; E)$$

defines a homomorphism-valued discrete 1-form.

Definition

The discrete connection 1-form is $A := \nabla - d_E \in C^1(X; \mathrm{Hom}(E))$. Here d_E is the trivial discrete connection. We use the notation

$$A_{ij} := \langle A,\ [i,j]\rangle_{i,j} = U_{ij} - \mathrm{Id}_{ij}$$

to denote the value of A on an edge.

Connection 1-cochains and gauge transformations

Definition

Given $B \in C^k(X; \mathrm{Hom}(E))$ and $D \in C^l(X; \mathrm{Hom}(E))$

$$\langle B \smile D,\ [0 \dots k+l] \rangle_{0,k+l} := \langle B,\ [0 \dots k] \rangle_{0,k}\ \langle D,\ [k \dots k+l] \rangle_{k,k+l}$$

where the right hand side is a composition of linear maps.

Lemma

Given discrete curvature $F_\nabla \in C^2(X; \mathrm{Hom}(E))$, and discrete connection 1-cochain $A \in C^1(X; \mathrm{Hom}(E))$, we have

$$F_\nabla = d_E A + A \smile A \in C^2(X; \mathrm{Hom}(E))$$

Connection 1-cochains and gauge transformations

Proposition

Given a gauge transformation $g \in C^0(X; \mathrm{Hom}(E))$ between discrete bundles with connection $(E, \nabla) \mapsto (E, \nabla')$, the corresponding connection 1-forms and curvature are related by

$$A \mapsto A' = g\,A\,g^{-1} - dg\,g^{-1}, \quad F' = gFg^{-1}.$$

Coarsening

Coarsened vector bundle and coarse discrete forms

Definition

For $X := \mathrm{sd}\,\tilde{X}$ and $(E, \nabla) \to X$ the data for the coarsened vector bundle with connection $(\tilde{E}, \widetilde{\nabla}) \to \tilde{X}$ has fibers $\tilde{E}_{c(\tilde{\sigma})}$ for $\tilde{\sigma} \in \tilde{X}$ and parallel transport maps $U_{c(\tilde{\sigma}^k), c(\tilde{\sigma}^{k+1})}$ for all $\tilde{\sigma}^k \prec \tilde{\sigma}^{k+1} \in \tilde{X}$.

Definition

For $\alpha \in C^k(X; E)$ and $\tilde{\sigma}^k \in \tilde{X}$ the discrete forms coarsening map $\mathfrak{C} : C^k(X; E) \to \tilde{C}^k(\tilde{X}; \tilde{E})$ is defined by

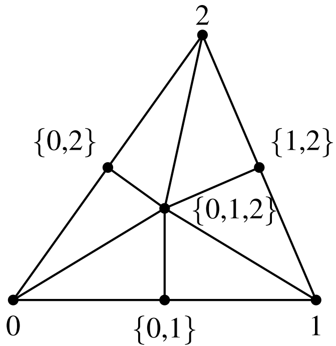
$$\langle \mathfrak{C}(\alpha),\ \tilde{\sigma}^k \rangle_{c(\tilde{\sigma}^k)} = \sum_{\sigma^k \in \mathrm{sd}\,\tilde{\sigma}^k \subset X} o(\sigma^k, \tilde{\sigma}^k)\ \langle \alpha,\ \sigma^k \rangle_{c(\tilde{\sigma}^k)}\,.$$

Why coarsening works for discrete vector bundle valued forms

Discrete forms coarsening map

For $\alpha \in C^k(X;E)$ and $\tilde{\sigma}^k \in \tilde{X}$:

$$\langle \mathfrak{C}(\alpha),\ \tilde{\sigma}^k \rangle_{c(\tilde{\sigma}^k)} = \sum_{\sigma^k \in \operatorname{sd} \tilde{\sigma}^k \subset X} o(\sigma^k, \tilde{\sigma}^k)\ \langle \alpha,\ \sigma^k \rangle_{c(\tilde{\sigma}^k)}\ .$$



Why coarsening works for discrete vector bundle valued forms

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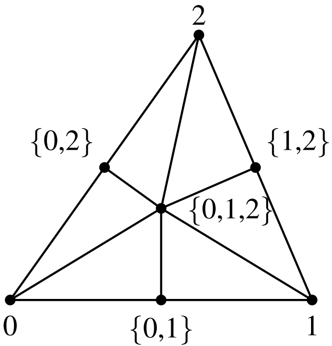

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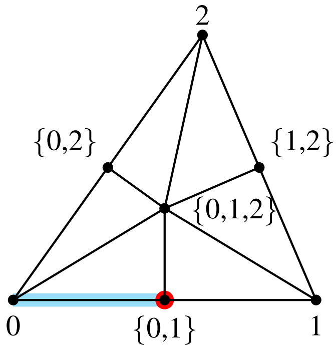

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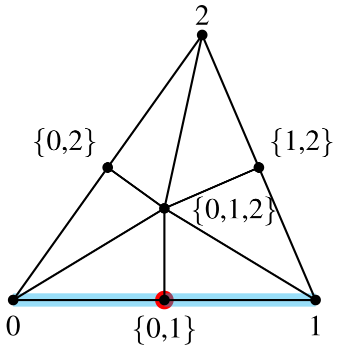

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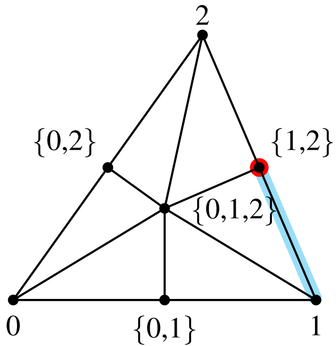

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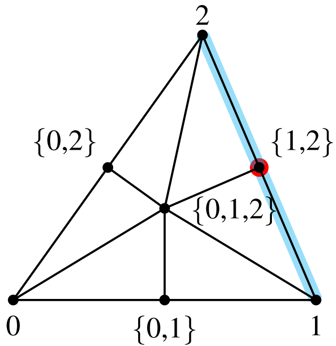

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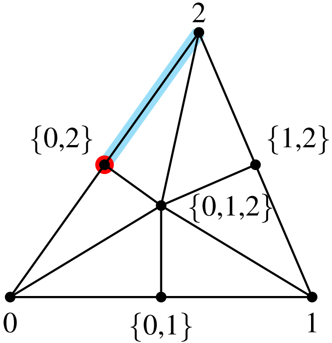

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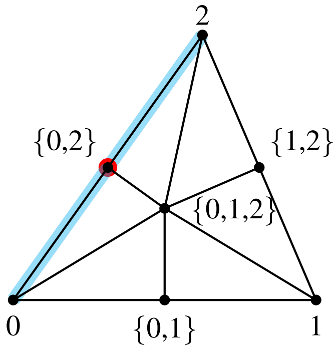

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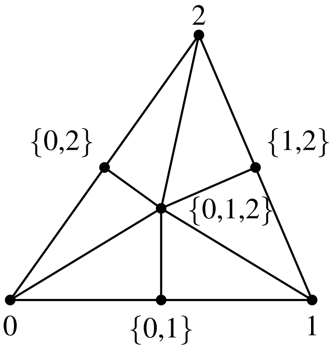

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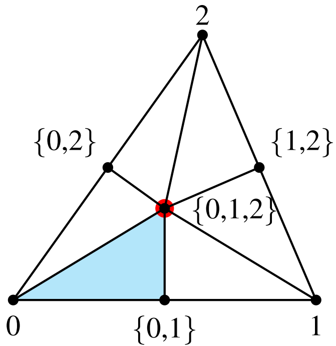

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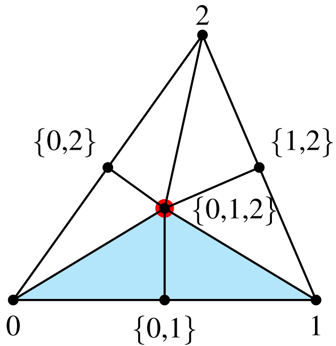

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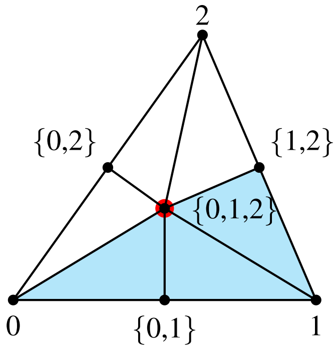

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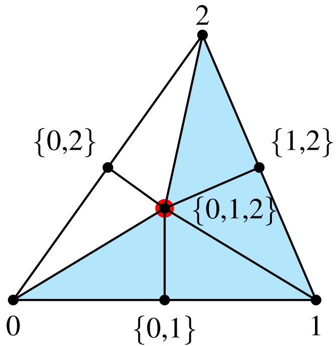

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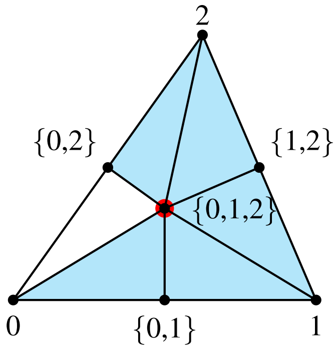

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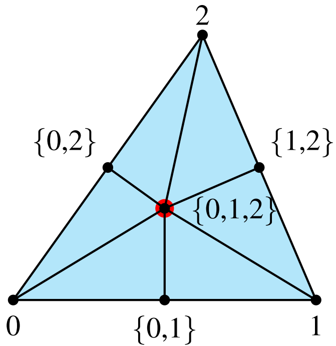

Coarse $d_{\widetilde{\nabla}}$

Definition

The coarse discrete exterior covariant derivative $d_{\widetilde{\nabla}} : \tilde{C}^k(\tilde{X};\tilde{E}) \longrightarrow \tilde{C}^{k+1}(\tilde{X};\tilde{E})$ is defined by its value on $\tilde{\sigma}^{k+1} \in \tilde{X}$ as

$$\langle d_{\widetilde{\nabla}}\tilde{\alpha},\ \tilde{\sigma}^{k+1}\rangle_{c(\tilde{\sigma}^{k+1})} = \sum_{\tilde{\sigma}^k \prec \tilde{\sigma}^{k+1} \in \tilde{X}} o(\tilde{\sigma}^k, \tilde{\sigma}^{k+1})\ U_{c(\tilde{\sigma}^{k+1}),c(\tilde{\sigma}^k)}\ \langle \tilde{\alpha},\ \tilde{\sigma}^k\rangle_{c(\tilde{\sigma}^k)}\,,$$

for $\tilde{\alpha} \in \tilde{C}^k(\tilde{X};\tilde{E})$. (For $k = 0$, the analogous formula defines the coarse connection $\widetilde{\nabla}$).

Proposition

The coarsening map commutes with the exterior covariant derivative, $d_{\widetilde{\nabla}}\mathfrak{C} = \mathfrak{C}d_{\nabla}$.

Coarse homomorphism valued discrete forms

Definition

For $\tilde{\sigma}^k, \tilde{\sigma}^{k+l} \in \tilde{X}$, $0 \le k \le \dim \tilde{X} - l$, a coarse homomorphism valued discrete l-form assigns to each l-dimensional combinatorial cube $D(\tilde{\sigma}^k \preceq \tilde{\sigma}^{k+l}) \in \mathcal{Q}(\tilde{X})$ a homomorphism $\tilde{E}_{c(\tilde{\sigma}^k)} \longrightarrow \tilde{E}_{c(\tilde{\sigma}^{k+l})}$.

Definition

For $B \in C^l(X; \mathrm{Hom}(E))$ and $\tilde{\sigma}^k \preceq \tilde{\sigma}^{k+l} \in \tilde{X}$, the homomorphism coarsening map $\mathfrak{C}^{\mathrm{Hom}} : C^l(X; \mathrm{Hom}(E)) \longrightarrow \tilde{C}^l(\mathcal{Q}(\tilde{X}); \mathrm{Hom}(\tilde{E}))$ is defined by

$$\left\langle \mathfrak{C}^{\mathrm{Hom}}(B),\ D(\tilde{\sigma}^k \preceq \tilde{\sigma}^{k+l}) \right\rangle_{c(\tilde{\sigma}^{k+l}), c(\tilde{\sigma}^k)} = \sum_{c[\tilde{\sigma}^{k+l} \succeq \tilde{\sigma}^k] \in \mathrm{sd}\, \tilde{X}} \mathrm{sgn}(c[\tilde{\sigma}^{k+l} \succeq \tilde{\sigma}^k]) \left\langle B,\ c[\tilde{\sigma}^{k+l} \succeq \tilde{\sigma}^k] \right\rangle_{c(\tilde{\sigma}^{k+l}), c(\tilde{\sigma}^k)}.$$

where $\mathrm{sgn}(c[\tilde{\sigma}^{k+l} \succeq \tilde{\sigma}^k]) = (-1)^{kl} o(c[\tilde{\sigma}^k \succeq \tilde{\sigma}^0], \tilde{\sigma}^k)\ o(c[\tilde{\sigma}^{k+l} \succeq \tilde{\sigma}^0], \tilde{\sigma}^{k+l})$

Coarse discrete curvature

Definition

The coarse discrete curvature of $\widetilde{\nabla}$, denoted $F_{\widetilde{\nabla}} \in C^2(Q(\tilde{X}); \mathrm{Hom}(\tilde{E}))$, evaluates on $D(\tilde{\sigma}^k \prec \tilde{\sigma}^{k+2})$ as a homomorphism $\tilde{E}_{c(\tilde{\sigma}^k)} \to \tilde{E}_{c(\tilde{\sigma}^{k+2})}$:

$$\begin{aligned}\left\langle F_{\widetilde{\nabla}},\ D(\tilde{\sigma}^k \prec \tilde{\sigma}^{k+2})\right\rangle_{c(\tilde{\sigma}^{k+2}),c(\tilde{\sigma}^k)} &:= \\ &\pm \left(U_{c(\tilde{\sigma}^{k+2}),\,c(\tilde{\sigma}_1^{k+1})}\, U_{c(\tilde{\sigma}_1^{k+1}),\,c(\tilde{\sigma}^k)} - U_{c(\tilde{\sigma}^{k+2}),\,c(\tilde{\sigma}_2^{k+1})}\, U_{c(\tilde{\sigma}_2^{k+1}),\,c(\tilde{\sigma}^k)}\right),\end{aligned}$$

where $\tilde{\sigma}^k \prec \tilde{\sigma}_1^{k+1}, \tilde{\sigma}_2^{k+1} \prec \tilde{\sigma}^{k+2}$.

Proposition

With $\tilde{X}, X = \mathrm{sd}\,\tilde{X}$, $(E, \nabla) \to X$ and $(\tilde{E}, \widetilde{\nabla}) \to \tilde{X}$ as above, coarse curvature $F_{\widetilde{\nabla}}$ is the coarsened fine curvature. That is, $F_{\widetilde{\nabla}} = \mathfrak{C}^{\mathrm{Hom}}(F_{\nabla})$.

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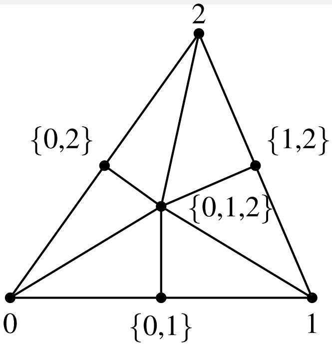

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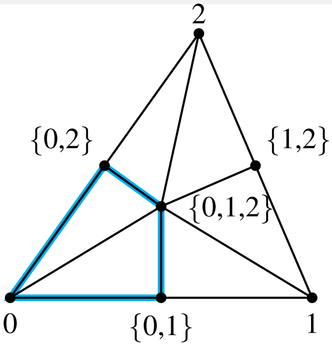

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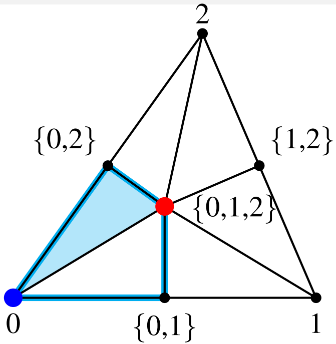

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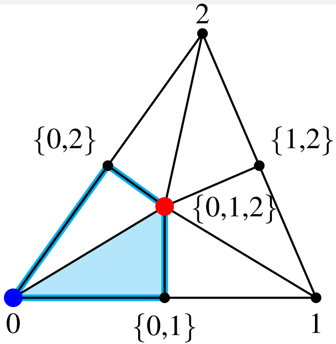

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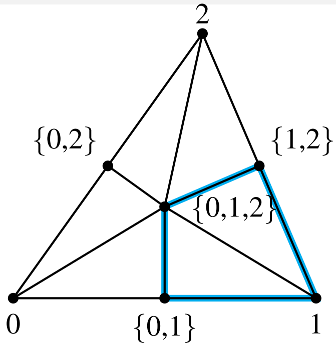

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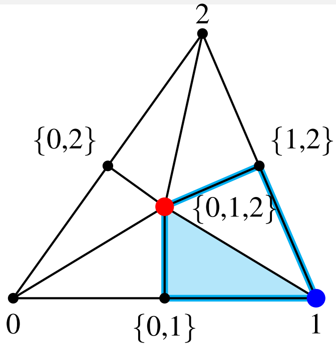

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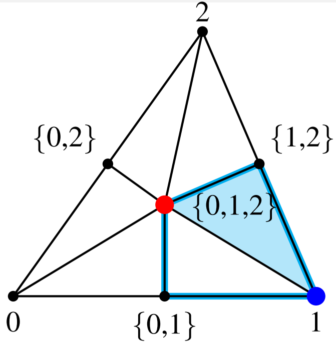

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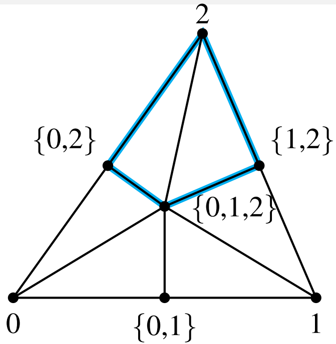

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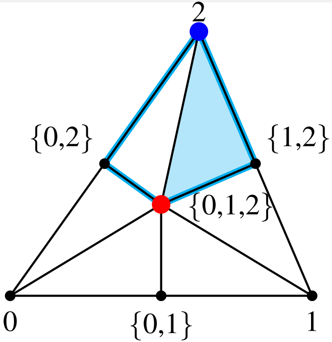

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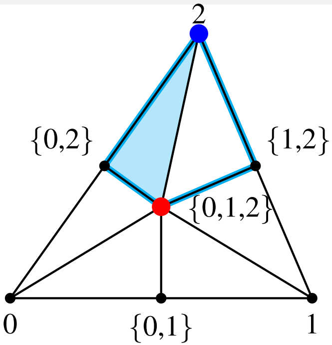


Other Constructions

Christiansen-Hu construction

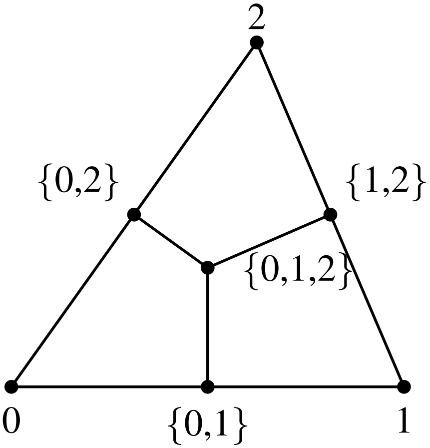
- Fibers are associated with simplices

Christiansen-Hu construction

- Fibers are associated with simplices
- Transport maps are between simplices differing in dimension by 1

Discrete exterior covariant derivative

Christiansen-Hu



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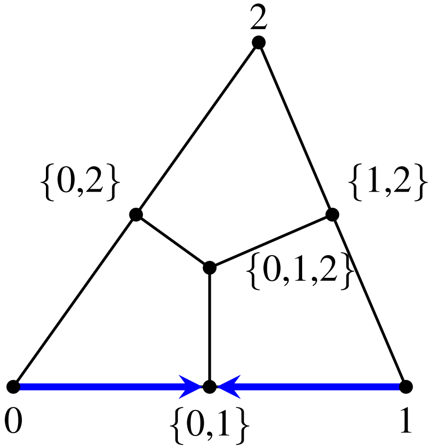

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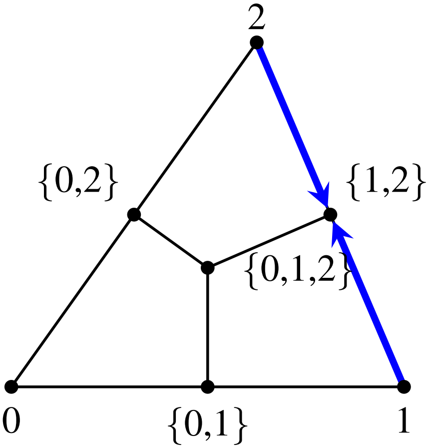

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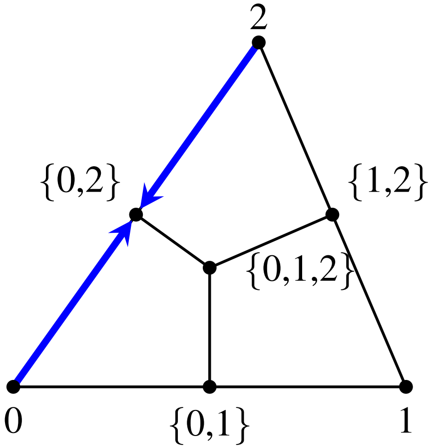

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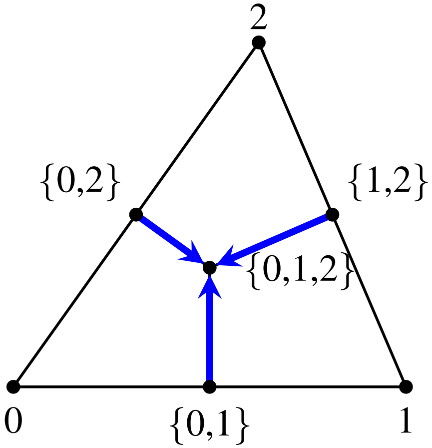

Braune, Tong, Gay-Balmaz, Desbrun construction

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  - Anti-symmetrization of our d_∇

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- A product for $C^k(X;\mathrm{Hom}(E)) \times C^\bullet(X;E)$ may exist for general k

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- See talk of Theo Braune tomorrow!

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- Explicit product for $C^k(X;\mathrm{Hom}(E)) \times C^l(X;E)$ for general k is not defined
- Implicitly defined product for $C^2(X;\mathrm{Hom}(E)) \times C^l(X;E)$
- Recent conjecture (?) $d_\nabla(F_\nabla \wedge \alpha) = F_\nabla \wedge d_\nabla \alpha$

Conclusions and outlook

- The three constructions are related
- Other two can be generated from ours
- Product limitations in the other two are puzzling
- Convergence may need to be studied in right norms (likely appropriate negative norms)
- For example the angle defect was shown to not converge in L^2 and H^{-1} norms [Gawlik 2020] and shown to converge in H^{-2} norm [Gawlik and Neunteufel 2025]